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Second-Order Fault Tolerant Extended Kalman Filter for Discrete Time Nonlinear Systems

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Abstract:

As missing sensor data may severely degrade the overall system performance and stability, reliable state estimation is of great importance in modern data-intensive control, computing, and power systems applications. Aiming at providing a more robust and resilient state estimation technique, this paper presents a novel second-order fault-tolerant extended Kalman filter estimation framework for discrete-time stochastic nonlinear systems under sensor failures, bounded observer-gain perturbation, extraneous noise, and external disturbances condition. The failure mechanism of multiple sensors is assumed to be independent of each other with various

malfunction rates. The proposed approach is a locally unbiased, minimum estimation error covariance based nonlinear observer designed for dynamic state estimation under these conditions. It has been successfully applied to a benchmark target-trajectory tracking application. Computer simulation studies have demonstrated that the proposed second-order fault-tolerant extended Kalman filter provides more accurate estimation results, in comparison with traditional first- and second-order extended Kalman filter. Experimental results have demonstrated that the proposed second-order fault-tolerant extended Kalman filter can serve as a powerful alternative to the existing nonlinear estimation approaches.

SECTION I. Introduction

Kalman filter, the extended Kalman filter, and the second-order extended Kalman filter have a wide range of industrial applications for dynamic estimation over the past 50 years. The original work of celebrated Kalman filter can be found in [1], [2]. The first-order extended Kalman filter, also known as the quasilinear Kalman filter, has been reported in [3]–[6]. Among them, the stochastic stability of extended Kalman filter has been investigated by Reif *et al.* [5] and [6]. The second-order extended Kalman filter was proposed in the paper written by Athens *et al.* [7].

To improve the robustness of nonlinear estimation, the H_∞ type of nonlinear filtering is developed, which guarantees the H_∞ norm of the mapping between estimation error and the extraneous disturbances is upper-bounded by a positive preset value. Different from the Kalman filtering approaches, H_∞ filters do not require any statistical properties of the external disturbances, but require the disturbances to be of $\mathcal{L}_2$ type. The first-order extended H_∞ filter was proposed by Berman and Shaked for continuous and discrete-time nonlinear systems in [8] and [9], respectively. Zhang *et al.* presented a stochastic H∞ estimation problem in [10], involving Itô stochastic differential equation and nonlinear Hamilton–Jacobi inequalities. The mixed H_2/H_∞discrete-time nonlinear filtering combines the advantages of quadratic optimality with the H_∞ type of disturbance rejection [11]–[16]. The second-order extended H_∞ filter for nonlinear discrete-time systems using quadratic error matrix approximation was studied by Hu and Yang [17].

It is not uncommon that sensor measurements do not contain accurate signals, but corrupted signals thanks to extraneous noise, external disturbances, sensor failures, delays, attenuations, distortions, multipaths, electromagnetic interference, etc. To address this issue, the problem of estimation with missing measurements was first pointed out by Nahi [18], Hadidi and Schwartz [19], in which the missing data were modeled by a binary switching sequence specified by a conditional probability distribution. Research in the area of linear filtering with missing measurements has been mushrooming during the past decades [20], [21]. Wang *et al.* considered the variance-constrained filtering problem for discrete-time stochastic systems with probabilistic missing measurements, subject to norm-bounded parameter uncertainties [20]. In [21], Wang *et al.* presented a robust finite-horizon estimator for linear system with missing measurements. Recent development involving the case of nonlinear systems with multiple sensors, which may malfunction independently, has been studied by Hounkpevi and Yaz in [22]. Hu *et al.* applied the extended Kalman filter to estimate the state variables of stochastic nonlinear systems with multiple missing measurement in [23].

The purpose of this paper is to present a second-order fault-tolerant estimator, which can be used to generate state estimates for discrete-time nonlinear dynamical system, from noisy observations of its outputs with faulty sensors, made in discrete instants of time. Leveraging our previous effort in [24] and [25], we propose a novel second-order, locally unbiased, minimum estimation error covariance based nonlinear observer, which is customized for state estimation under sensor failures, bounded observer gain perturbation, and external disturbances. Hence, the filter proposed herein shall be referred as the second-order fault-tolerant extended Kalman filter.

The paper is organized as follows: First, the system and measurement model formulation is investigated in Section II. Then, the structure of the second-order fault-tolerant extended Kalman filter (second-order FTEKF) is derived in Section III. After that, Section IV presents special cases, which are the second-order fault-tolerant extended Kalman filter without estimator-gain uncertainties, and the second-order extended Kalman filter. Application to a benchmark problem involving reconstructing the trajectory of a target using the recorded range measurements is discussed in Section V. In this section, comparisons of the first-, second-order extended Kalman filter (EKF), and the second-order fault tolerant extended Kalman filter (FTEKF) nonlinear estimation are illustrated through computer simulation studies. Finally, conclusion is reached in Section VI.

The following standard notation is used in this work: $x \in \mathcal{R}^n$denotes n-dimensional real vector with norm $\|x\| = (x^T x)^{1/2}$where $(\cdot)^T$ indicates matrix transpose. $A \geq 0$ for a symmetric matrix denotes a positive semidefinite matrix. P denotes the covariance matrix. $\overline{x}$ is the mean value for x. $Prob(\cdot)$is the probability of an event. $E\{x\} = \overline{x}$ is the mean/expectation value of a random variable x. $x \sim (\overline{x}, X)$ denotes a random variable x with arbitrary distribution with mean $\overline{x}$ and covariance$X$. δ_{k-j} is the Kronecker delta function; that is, $\delta_{k-j} = 1$ when $k = j$; and $\delta_{k-j} = 0$ when $k \neq j$. Let A and B be $n \times m$ matrices, the Hadamard product of A and B is denoted by $A \circ B$, and is defined as $[A \circ B]_{i,j} = [A]_{ij}[B]_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq m$. Matrix form of Rayleigh's inequality is also used in the derivation of this work, which may be stated as: for matrices $X = X^T \in \mathcal{R}^{n \times n}$and $Y \in \mathcal{R}^{m \times n}$, the matrix inequality $\lambda_{\min}(X)YY^T \leq YXY^T \leq \lambda_{\max}(X)YY^T$ holds. $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$stand for the minimum and maximum eigenvalues of a matrix. $\mathrm{tr}\{\cdot\}$denotes the trace of a matrix.

SECTION II. Structure of the Plant

Consider the following discrete-time nonlinear stochastic system:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + v_k \\ y_k &= \begin{pmatrix} \gamma_k^1 h_1(x_k, u_k) + w_k^1 \\ \gamma_k^2 h_2(x_k, u_k) + w_k^2 \\ \vdots \\ \gamma_k^p h_p(x_k, u_k) + w_k^p \end{pmatrix} \end{aligned} \quad (1)$$

where

$x_k \in \mathcal{R}^n$ state space variable;
$u_k \in \mathcal{R}^m$ control input;
$v_k \in \mathcal{R}^n$ process disturbance and perturbation;
$y_k \in \mathcal{R}^p$ measurement output;
$w_k^i \in \mathcal{R}$ measurement disturbance in each sensor;
f, h nonlinear process and measurement equations.

The mean of initial state x_0 is $E\{x_0\} = \overline{x}_0$ and covariance $X_0 = E\{(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T\}$. The noise processes, v_k and w_k, are white, zero mean, uncorrelated with each other and with x_0, and have covariance V_k and W_k, respectively

$$\begin{aligned} v_k &\sim (0, V_k), w_k \sim (0, W_k) \\ E\{v_k v_j^T\} &= V_k \delta_{k-j}, E\{w_k w_j^T\} = W_k \delta_{k-j} \\ E\{v_k w_j^T\} &= 0, E\{v_k x_0^T\} = 0, E\{w_k x_0^T\} = 0. \end{aligned} \quad (2)(3)$$

The scalar binary Bernoulli distributed random variables γ_k^i are with mean π_i and variance $\pi_i(1 - \pi_i)$, whose possible outcomes ({0,1}) are defined as $P(\gamma_k^i = 1) = \pi_i$ and $P(\gamma_k^i = 0) = 1 - \pi_i$. The formulation involves hard sensor failures, where the sensor either works perfectly or it fails totally. There is no other alternative considered in this work.

To simplify (1), let us denote the sensing condition matrix as

$$\Gamma_k = \mathrm{diag}\left[\gamma_k^1, \gamma_k^2, \dots, \gamma_k^p\right]. \quad (4)$$

Denote the measurement dynamics matrix as

$$h(x_k, u_k) = [h_1(x_k, u_k), h_2(x_k, u_k), \dots, h_p(x_k, u_k)]^T. \quad (5)$$

Denote the extraneous measurement noise vector as

$$w_k = [w_k^1, w_k^2, \dots, w_k^p]^T. \quad (6)$$

Hence, the measurement equation can be written as

$$y_k = \Gamma_k h(x_k, u_k) + w_k. \quad (7)$$

Neglect the higher-order terms, the second-order Taylor series expansions of $f(x_k, u_k), h(x_k, u_k)$ around the estimated state $\hat{x}_k$ can be expressed as

$$\begin{aligned} f(x_k, u_k) \approx\ & f\left(\hat{x}_k, u_k\right) + \frac{\partial f}{\partial x_k}\Big|_{x_k=\hat{x}_k}\left(x_k - \hat{x}_k\right) \\ & + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f \cdot (x_k - \hat{x}_k)^T \frac{\partial^2 f_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k) \end{aligned} \quad (8)$$

and

$$\begin{aligned} h(x_k, u_k) \approx\ & h(\hat{x}_k, u_k) + \frac{\partial h}{\partial x_k}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k) \\ & + \frac{1}{2}\sum_{i=1}^{p} \phi_i^h \cdot (x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k) \end{aligned} \quad (9)$$

where f_i and h_i are the ith element of $f(x_k, u_k)$ and $h(x_k, u_k)$, respectively. ϕ_i^f and ϕ_i^h are $n \times 1$ and $p \times 1$ column vectors, respectively. ϕ_i denotes a column vector with all zeros except for a one in the ith element, i.e.,

$$\phi_i = [0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0]^T. \quad (10)$$

The quadratic terms in (8) and (9) can be written as

$$\begin{aligned} & (x_k - \hat{x}_k)^T \frac{\partial^2 f_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k) \\ & = \mathrm{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right\} \end{aligned} \quad (11)$$

and

$$\begin{aligned} & (x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k) \\ & = \mathrm{tr}\left\{\frac{\partial^2 h_i}{\partial x_k^2}\Big|_{x_k=\hat{x}_k}(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right\}. \end{aligned} \quad (12)$$

By denoting the estimation error covariance matrix P_k as

$$P_k = E\left\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right\} \quad (13)$$

(11) and (12) can be approximated as

$$(x_k - \hat{x}_k)^T \frac{\partial^2 f_i}{\partial x_k^2}(x_k - \hat{x}_k) \approx \mathrm{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\} \quad (14)$$

and

$$(x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}(x_k - \hat{x}_k) \approx \mathrm{tr}\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}. \quad (15)$$

Hence, if we evaluate (8) at $x_k = \hat{x}_k$ and substitute (14) in the summation, we have

$$\begin{aligned} f(x_k, u_k) \approx \quad & f(\hat{x}_k, u_k) + \frac{\partial f}{\partial x_k}|_{x_k = \hat{x}_k}(x_k - \hat{x}_k) \\ & + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f \,\mathrm{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}. \end{aligned} \quad (16)$$

Likewise, if we evaluate (9) at $x_k = \hat{x}_k$ and substitute (15) in the summation, we have

$$\begin{aligned} h(x_k, u_k) \approx \quad & h(\hat{x}_k, u_k) + \frac{\partial h}{\partial x_k}|_{x_k = \hat{x}_k}(x_k - \hat{x}_k) \\ & + \frac{1}{2}\sum_{i=1}^{p} \phi_i^h \mathrm{tr}\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}. \end{aligned} \quad (17)$$

SECTION III. Structure of the Second-Order Fault-Tolerant Extended Kalman Filter

The purpose of this novel second-order fault-tolerant nonlinear filter is to estimate the state vector x_k based on the knowledge of system dynamics and the availability of noisy measurement y_k under the effect of random sensor failures, bounded observer-gain perturbation, external disturbances, and extraneous noise.

The objective is to minimize the cost function defined as

$$J = \min_{K_k} E\left\{\sum_{k=0}^{N} ||x_k - \hat{x}_k||_2^2\right\} \quad (18)$$

subject to

$$E\{x_k - \hat{x}_k\} = 0. \quad (19)$$

Motivated by the one-step Kalman filter form in [26], the following discrete-time nonlinear Luenberger-type observer is adopted in this paper

$$\begin{aligned} \hat{x}_{k+1} = \quad & f(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f \,\mathrm{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\} \\ & + (K_k + \Delta_k)[y_k - \overline{\Gamma}_k h(\hat{x}_k, u_k)] - \varrho_k. \end{aligned} \quad (20)$$

To develop a resilient nonlinear estimator against observer gain perturbation, Kalman gain $K_k + \Delta_k$ is utilized in (20). Though the Kalman filter gain should be K_k, due to computational or tuning uncertainties Δ_k, it is erroneously implemented as $K_k + \Delta_k$.

The term $\overline{\Gamma}_k$, the reliability expectation matrix of p independent sensors, can be defined as

$\overline{\Gamma}_k = E\{\Gamma_k\} = \text{diag}[\pi_1, \pi_2, \ldots, \pi_p]$ (21)

with π_i is the probability of the ith sensor to work perfectly, i.e., being an accurate sensor to provide reliable measurements.

The predictor part of (20) is a replica of the nonlinear plant dynamics. The correction term corrects future state estimates based on the present error in estimation of the measured value. The vector ϱ_k is the output bias correction term. The choice of ϱ_k makes $\hat{x}_{k+1}$an unbiased estimate of x_{k+1}.

K_k is the feedback gain with additive uncertainty Δ_k. The uncertainty, Δ_k, is assumed to have zero mean, bounded second moment, and be uncorrelated with initial state, process, and measurement noises, i.e.,

$$\begin{aligned} E\{\Delta_k\Delta_k^T\} &\leq \delta I, E\{\Delta_k^T x_0\} = 0 \\ E\{\Delta_k^T v_k\} &= 0, E\{\Delta_k w_k\} = 0 \end{aligned} \quad (22)$$

where the upper-bound δ is a positive constant.

III. Theorem 1:

The Second-Order Fault-Tolerant Extended Kalman Filter

The second-order fault-tolerant extended Kalman filter is initialized by

$$\begin{aligned} \hat{x}_0 &= E\{x_0\} \\ P_0 &= E\{(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\}. \end{aligned} .(23)$$

By computing the Jacobian matrices

$$A_k = \frac{\partial f}{\partial x}|_{x=\hat{x}_k}, C_k = \frac{\partial h}{\partial x}|_{x=\hat{x}_k}.(24)$$

For time steps $k = 1,2,3, \ldots$, denote

$$\zeta(\hat{x}_k) = h(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{p} \phi_i^h tr\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{x_k=\hat{x}_k} P_k\right\}.(25)$$

The nonlinear estimator propagates by computing the Kalman filter gain

$$\begin{aligned} K_k^o = \ & (A_k P_k C_k^T \overline{\Gamma}_k^T)[\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T \\ & +\Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T) + W_k]^{-1}. \end{aligned} \quad (26)$$

The upper-bound of estimation error covariance is updated through

$$\begin{aligned} P_{k+1} = \ & A_k P_k A_k^T + V_k + \lambda_{\max}\{\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T + W_k \\ & +\Upsilon \circ [\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T]\}\delta I \\ & -(A_k P_k C_k^T \overline{\Gamma}_k^T)\{\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T \\ & +\Upsilon \circ [\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T] + W_k\}^{-1}(\overline{\Gamma}_k C_k P_k A_k^T). \end{aligned} \quad (27)$$

The state estimate is updated through

$$\hat{x}_{k+1} = f(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f tr\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\} + (K_k^o + \Delta_k)[y_k - \overline{\Gamma}_k h(\hat{x}_k, u_k) - \overline{\Gamma}_k \frac{1}{2}\sum_{i=1}^{p} \phi_i^h tr\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}] \quad (28)$$

where

$$\Upsilon = \text{diag}[\pi_1(1-\pi_1), \pi_2(1-\pi_2), \dots, \pi_p(1-\pi_p)] = \begin{pmatrix} \pi_1(1-\pi_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \pi_p(1-\pi_p) \end{pmatrix}. \quad (29)$$

III. Proof:

By denoting the estimation error $e_k = x_k - \hat{x}_k$, and applying Taylor series expansion results in (8) and (9), we have

$$\begin{aligned} e_{k+1} = {} & x_{k+1} - \hat{x}_{k+1} = f(x_k, u_k) + v_k - \{f(\hat{x}_k, u_k) \\ & + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f tr\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{x_k=\hat{x}_k} P_k\right\} \\ & + (K_k + \Delta_k)[y_k - \overline{\Gamma}_k h(\hat{x}_k, u_k)]\} + \varrho_k \\ = {} & A_k e_k + v_k - (K_k + \Delta_k)\{\Gamma_k[h(\hat{x}_k, u_k) + C_k e_k)] \\ & + \Gamma_k \frac{1}{2}\sum_{i=1}^{p} \phi_i^h \cdot (x_k - \hat{x}_k)^T \frac{\partial^2 h_i}{\partial x_k^2}|_{x_k=\hat{x}_k} (x_k - \hat{x}_k) \\ & + w_k - \overline{\Gamma}_k h(\hat{x}_k, u_k)\} + \varrho_k + \mathcal{O}(e_k^2). \end{aligned} \quad (30)$$

Apply constraint (19) to (30), we demand that $\hat{x}_{k+1}$is an unbiased estimate of x_{k+1}. The following choice of ϱ_k makes $\hat{x}_{k+1}$ an unbiased estimate

$$\varrho = (K_k + \Delta_k)\overline{\Gamma}_k \frac{1}{2}\sum_{i=1}^{p} \phi_i^h tr\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{x_k=\hat{x}_k} P_k\right\}. \quad (31)$$

By neglecting the higher-order error term $\mathcal{O}(e_k^2)$, and applying (17), we have

$$\begin{aligned} e_{k+1} \approx {} & [A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k + v_k - (K_k + \Delta_k)w_k \\ & - (K_k + \Delta_k)\tilde{\Gamma}_k[h(\hat{x}_k, u_k) \\ & + \frac{1}{2}\sum_{i=1}^{p} \phi_i^h tr\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{x_k=\hat{x}_k} P_k\right\}] \end{aligned} \quad (32)$$

where

$$\tilde{\Gamma}_k = \Gamma_k - \overline{\Gamma}_k. \quad (33)$$

To simplify (32) in the derivation process, let us denote

$$\zeta(\hat{x}_k) = h(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{p} \phi_i^h tr\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{x_k=\hat{x}_k} P_k\right\}. \quad (34)$$

To derive the optimal estimator gain K_k, applying (32), the estimation error covariance matrix evolves as

$$
\begin{aligned}
P_{k+1} = \quad & E\{e_{k+1}e_{k+1}^T\} \\
= \quad & E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} \\
& + E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k v_k^T\} \\
& + E\{v_k e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} \\
& - E\{(A_k - (K_k + \Delta_k)\Gamma_k C_k)e_k w_k^T (K_k + \Delta_k)^T\} \\
& - E\{(K_k + \Delta_k)w_k e_k^T (A_k - (K_k + \Delta_k)\Gamma_k C_k)^T\} \\
& - E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T (K_k + \Delta_k)^T\} \\
& - E\{(K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k) e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} \\
& + V_k + E\{(K_k + \Delta_k)W_k (K_k + \Delta_k)^T\} \\
& - E\{v_k w_k^T (K_k + \Delta_k)^T + (K_k + \Delta_k) w_k v_k^T\} \\
& - E\{v_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T (K_k + \Delta_k)^T + (K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k) v_k^T\} \\
& + E\{(K_k + \Delta_k) w_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T (K_k + \Delta_k)^T \\
& + (K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k) w_k^T (K_k + \Delta_k)^T\} \\
& + E\{(K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T (K_k + \Delta_k)^T\}.
\end{aligned} \tag{35}
$$

Each individual term of (35) can be reduced as follows: Applying (22) and Rayleigh's matrix inequality, the first term can be simplified to

$$
\begin{aligned}
& E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} \\
& \leq [A_k - K_k \overline{\Gamma}_k C_k] P_k [A_k - K_k \overline{\Gamma}_k C_k]^T \\
& + K_k E\{\tilde{\Gamma}_k C_k P_k C_k^T \tilde{\Gamma}_k\} K_k^T \\
& + \lambda_{\max}(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T + E\{\tilde{\Gamma}_k C_k P_k C_k^T \tilde{\Gamma}_k^T\})\delta I.
\end{aligned} \tag{36}
$$

Since e_k and v_k are uncorrelated, we have

$$
\begin{aligned}
& E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k v_k^T\} \\
& + E\{v_k e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} = 0.
\end{aligned} \tag{37}
$$

Since e_k and w_k are uncorrelated, we have

$$
\begin{aligned}
& E\{(A_k - (K_k + \Delta_k)\Gamma_k C_k)e_k w_k^T (K_k + \Delta_k)^T\} \\
& + E\{(K_k + \Delta_k)w_k e_k^T (A_k - (K_k + \Delta_k)\Gamma_k C_k)^T\} = 0.
\end{aligned}
$$

Since $e_k, \Delta_k, \tilde{\Gamma}_k$ are mutually uncorrelated and $E\{e_k\} = 0, E\{\tilde{\Gamma}_k\} = 0$, the following term is zero as shown below

$$
\begin{aligned}
& E\{[A_k - (K_k + \Delta_k)\Gamma_k C_k]e_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T (K_k + \Delta_k)^T\} \\
& + E\{(K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k) e_k^T [A_k - (K_k + \Delta_k)\Gamma_k C_k]^T\} = 0.
\end{aligned} \tag{38}
$$

Applying (22), it is easy to show that

$$
\begin{aligned}
& V_k + E\{(K_k + \Delta_k)W_k (K_k + \Delta_k)^T\} \\
& \leq V_k + K_k W_k K_k^T + \lambda_{\max}(W_k)\delta I.
\end{aligned} \tag{39}
$$

Since $v_k, w_k$are uncorrelated, the term is reduced to

$$E\{v_k w_k^T (K_k + \Delta_k)^T + (K_k + \Delta_k) w_k v_k^T\} = 0. \tag{40}$$

Since v_k, Δ_k are uncorrelated, and $E\{v_k\} = 0$, we have

$$E\{v_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T[K_k + \Delta_k]^T \\ -[K_k + \Delta_k]\tilde{\Gamma}_k \zeta(\hat{x}_k)v_k^T\} = 0. \quad (41)$$

Since $v_k, \Delta_k, \tilde{\Gamma}_k$ are mutually uncorrelated, and $E\{w_k\} = 0, E\{\tilde{\Gamma}_k\} = 0$, the term is zero as follows:

$$E\{(K_k + \Delta_k)w_k \zeta^T(\hat{x}_k)\tilde{\Gamma}_k^T(K_k + \Delta_k)^T \\ +(K_k + \Delta_k)^T\tilde{\Gamma}_k \zeta(\hat{x}_k)w_k^T(K_k + \Delta_k)\} = 0. \quad (42)$$

Applying (22), the term has an upper-bound as shown below

$$E\{(K_k + \Delta_k)\tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta(\hat{x}_k)^T\tilde{\Gamma}_k^T(K_k + \Delta_k)^T\} \\ \le K_k E\{\tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta(\hat{x}_k)^T\tilde{\Gamma}_k^T\}K_k^T \\ +\lambda_{\max}(E\{\tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta(\hat{x}_k)^T\tilde{\Gamma}_k^T\})\delta I. \quad (43)$$

Hence, based on the aforementioned reductions in (36)–(43), (35) yields

$$P_{k+1} \le [A_k - K_k\bar{\Gamma}_k C_k]P_k[A_k - K_k\bar{\Gamma}_k C_k]^T \\ +K_k E\{\tilde{\Gamma}_k C_k P_k C_k^T\tilde{\Gamma}_k^T + \tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta(\hat{x}_k)^T\tilde{\Gamma}_k^T + W_k\}K_k^T \\ +\lambda_{\max}(\bar{\Gamma}_k C_k P_k C_k^T\bar{\Gamma}_k^T + W_k \\ +E\{\tilde{\Gamma}_k C_k P_k C_k^T\tilde{\Gamma}_k^T + \tilde{\Gamma}_k \zeta(\hat{x}_k)\zeta(\hat{x}_k)^T\tilde{\Gamma}_k^T\})\delta I + V_k.$$

Since we have

$$E\{[\tilde{\Gamma}_k \zeta(\hat{x}_k)][\tilde{\Gamma}_k \zeta(\hat{x}_k)]^T + [\tilde{\Gamma}_k C_k]P_k[\tilde{\Gamma}_k C_k]^T\} \\ = \Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T)$$

where Υ is defined in (29), the upper bound on the error covariance equation can be obtained as

$$P_{k+1} = [A_k - K_k\bar{\Gamma}_k C_k]P_k[A_k - K_k\bar{\Gamma}_k C_k]^T + V_k \\ +K_k[W_k + \Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T)]K_k^T \\ +\lambda_{\max}\{\bar{\Gamma}_k C_k P_k C_k^T\bar{\Gamma}_k^T + W_k \\ +\Upsilon \circ [\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T]\}\delta I.$$

δI.

Equivalently, it can be organized as

$$P_{k+1} = \Omega_k + K_k\Lambda_k + \Lambda_k^T K_k^T + K_k\Phi_k K_k^T \quad (47)$$

where

$$\begin{aligned} \Phi_k &= \bar{\Gamma}_k C_k P_k C_k^k \bar{\Gamma}_k^T + \Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T) + W_k \\ \Lambda_k &= -\bar{\Gamma}_k C_k P_k A_k^T \\ \Omega_k &= A_k P_k A_k^T + V_k + \lambda_{\max}\{\bar{\Gamma}_k C_k P_k C_k^T\bar{\Gamma}_k^T + W_k + \\ &\quad \Upsilon \circ [\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T]\}\delta I. \end{aligned} \quad (48)$$

Applying completing the square in observer gain Kk

$$P_{k+1} = \Omega_k + (K_k - K_k^o)\Phi_k(K_k - K_k^o)^T - K_k^o\Phi_k K_k^{oT}. \quad (49)$$

For (44) to be equal to (46), the following condition must hold:

$K_k \Lambda_k = -K_k \Phi_k K_k^{oT}$.(50)

Therefore, the robust optimal feedback gain

$$K_k^o = \begin{array}{l} -\Lambda_k^T \Phi_k^{-1} = A_k P_k C_k^T \overline{\Gamma}_k^T \{\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T \\ + \Upsilon \circ [\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T] + W_k\}^{-1}. \end{array} \quad (51)$$

By setting $K_k = K_k^o$, the resulting matrix difference equation for the minimum of the upper bound on the estimation error covariance is given as $P_{k+1} = \Omega_k - K_k^o \Phi_k K_k^{oT}$, which leads to (27). This concludes the proof of Theorem 1.■

SECTION IV. Special Cases

A. Second-Order Fault-Tolerant Extended Kalman Filter Without Estimator Gain Uncertainties

Consider an extreme case, when we neglect perturbations on the estimator gain, i.e., $\delta = 0$, then it is easy to derive the second-order fault-tolerant extended Kalman filter without estimator gain, as follows.

The optimal Kalman observer gain is

$$K_k^o = \begin{array}{l} (AP_k C_k^T \overline{\Gamma}_k^T)[\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T \\ + \Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T) + W_k]^{-1}. \end{array} \quad (52)$$

The upper-bound on minimum estimation error covariance is given as

$$P_{k+1} = \begin{array}{l} A_k P_k A_k^T + V_k - (A_k P_k C_k^T \overline{\Gamma}_k^T)[\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T \\ + \Upsilon \circ (\zeta(\hat{x}_k)\zeta^T(\hat{x}_k) + C_k P_k C_k^T) + W_k]^{-1}(\overline{\Gamma}_k C_k P_k A_k^T) \end{array} \quad (53)$$

and the state estimate can be updated as follows:

$$\hat{x}_{k+1} = \begin{array}{l} f(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f \operatorname{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\} \\ + K_k^o \left[y_k - \overline{\Gamma}_k h(\hat{x}_k, u_k) - \frac{1}{2}\sum_{i=1}^{p} \phi_i^h \operatorname{tr}\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}\right]. \end{array} \quad (54)$$

B. Second-Order Extended Kalman Filter

Consider a more extreme case, when we further neglect the effect of sensor faults, i.e., $\gamma_k^i = 1$ for all sensors, then reliability matrix $\overline{\Gamma}_k$ becomes an identity matrix, and Υ matrix reduces to a zero matrix; the one-step second-order extended Kalman filter is obtained as a limiting case of the proposed second-order fault-tolerant extended Kalman filter in the following form.

The Kalman observer gain can be written as

$K_k = (A_k P_k C_k^T)[C_k P_k C_k^T + W_k]^{-1}$. (55)

The upper-bound on the minimum estimation error covariance equation can be found as

$$P_{k+1} = \begin{array}{l} A_k P_k A_k^T + V_k \\ -(A_k P_k C_k^T)[C_k P_k C_k^T + W_k]^{-1}(C_k P_k A_k^T) \end{array} \quad (56)$$

and the state estimate is updated in the form of

$$\hat{x}_{k+1} = f(\hat{x}_k, u_k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i^f \operatorname{tr}\left\{\frac{\partial^2 f_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\} + K_k\left[y_k - h(\hat{x}_k, u_k) - \frac{1}{2}\sum_{i=1}^{p} \phi_i^h \operatorname{tr}\left\{\frac{\partial^2 h_i}{\partial x_k^2}|_{\hat{x}_k} P_k\right\}\right]. \quad (57)$$

SECTION V. Applications to Benchmark Nonlinear Filtering Problem

This section presents the application of our novel second-order fault-tolerant extended Kalman filter to the problem of reconstructing the trajectory of a target using the recorded range measurements from a range-measuring device. Since its initial introduction in [7], this benchmark problem has been widely known and extensively used for examining the performance of nonlinear estimators [27], [28]. The purpose of this model is to estimate the trajectory of an object falling through an exponential atmosphere with a constant, yet unknown, drag coefficient. Measurement is the spatial distance from the range measuring device to the target. The system dynamics are summarized below [7], [27], [28]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{C_D A\rho}{2m} x_2^2 - g \end{aligned} \quad (58)(59)$$

where C_D is the air drag coefficient; A is the cross-sectional area; g is the gravitational acceleration; and ρ is the atmospheric density, which is governed by the following dynamics:

$$\rho = \rho_0 e^{-\eta x_1} = \rho_0 e^{-x_1/\kappa}. \quad (60)$$

The parameter $\eta = 1/\kappa$ is known as the constant inverse density scale height.

If the aerodynamics of the target are unknown, a third state variable, constant ballistic coefficient, can be setup to estimate in real time in order to characterize the dynamics of the target. This third state variable can be expressed as a positive constant from (56)

$$x_3 = \frac{C_D A\rho}{m} > 0. \quad (61)$$

The measuring device itself is located at an altitude of a, and at a horizontal distance b from the target's vertical line of fall. The range measuring device measures the spatial distance of this falling object as

$$h(x) = \sqrt{b^2 + (x_1 - a)^2}. \quad (62)$$

Hence, by including the process and measurement noise, and considering the effect of measurement failures, the complete system dynamics can be reached from (56), (58), and (59) as

$$\begin{aligned} \dot{x}_1 &= x_2 + v_1 \\ \dot{x}_2 &= \frac{1}{2}\rho_0 e^{-x_1/\kappa} x_2^2 x_3 - g + v_2 \\ \dot{x}_3 &= v_3 \\ y &= \gamma\sqrt{b^2 + (x_1 - a)^2} + w \end{aligned} \quad (63)(64)(65)(66)$$

where the process noise $v = [v_1 v_2 v_3]^T$ and the measurement noise w are mutually independent with zero mean Gaussian probability distributions. Due to external disturbance, the range measurement device may provide corrupted measurement, i.e., bad data. In measurement equation (63), γ is the variable to characterize the reliable measurement rate, while the mean value of γ is $\pi = 0.95$. In another word, the Bernoulli distributed measure failure rate is $1-\gamma$.

The Jacobian matrices can be derived by taking partial derivatives as follows:

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{(67)}$$

with

$$\begin{aligned} A_{21} &= -\tfrac{1}{2k}\rho_0 e^{-x_1/\kappa} x_2^2 x_3 \\ A_{22} &= \rho_0 e^{-x_1/\kappa} x_2 x_3 \qquad \text{(68)} \\ A_{23} &= \tfrac{1}{2}\rho_0 e^{-x_1/\kappa} x_2^2 \end{aligned}$$

and

$$C = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{x_1 - a}{\sqrt{b^2 + (x_1 - a)^2}} & 0 & 0 \end{bmatrix}. \text{(69)}$$

The Hessian matrices can be found as follows:

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x^2} &= \frac{\partial^2 f_3}{\partial x^2} = 0_{3\times 3} \\ \frac{\partial^2 f_2}{\partial x^2} &= \rho_0 e^{-x_1/\kappa} \begin{bmatrix} \frac{x_2^2 x_3}{2\kappa^2} & -x_2 x_3/\kappa & -\frac{x_2^2}{2\kappa} \\ -x_2 x_3/\kappa & x_3 & x_2 \\ -\frac{x_2^2}{2\kappa} & x_2 & 0 \end{bmatrix} \text{(70)} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} &= \begin{bmatrix} h^{-1}[1-(x_1-a)^2 h^{-2}] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{[b^2+(x_1-a)^2]^{1/2}} - \frac{(x_1-a)^2}{[b^2+(x_1-a)^2]^{3/2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \text{(71)}$$

The parameters and initial values used for conducting computer simulation studies are summarized in Table I.

TABLE I Model Parameter Values and Initial Values for Performing Nonlinear Estimation

Item	Unit	Value
a	ft	200
b	ft	10^5
κ	ft	$2 X 10^4$
g	ft/s^2	32.2
ρ_0	$lb - sec^2/ft^4$	2
$E\{v_i^2\}$	ft^2	0
$E\{w^2\}$	ft^2	100
γ		0.95
x_0		$[100000 \; -6000 \; 1/2000]^T$
$\hat{x}_0$		$[100100 \; -6100 \; 1/2600]^T$
P_0		$diag\{[500, 20000, \; 1/250000]^T\}$

Applying the Euler's discretization, the discrete time system model is derived with sampling period of $T = 0.01$ s. The simulation results are summarized as follows: The measurement y with sensor failures is shown in Fig. 1.

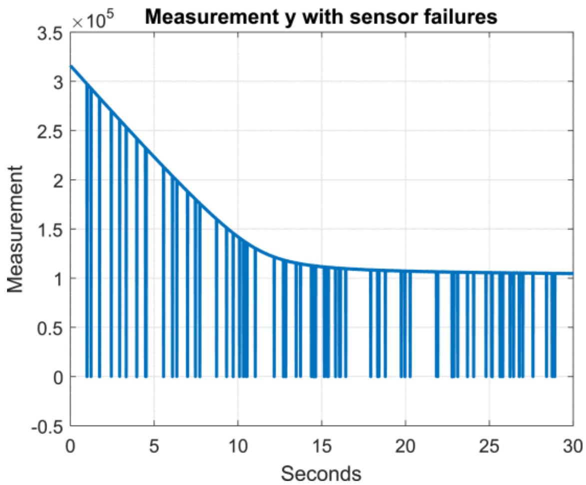

Fig. 1. Measurements with sensor failures when π=0.97.

The comparisons of true and estimated altitude values using the first-order EKF, the second-order EKF, and the proposed second-order FTEKF are shown in Fig. 2. Notice that under sensor failure conditions, the second-order fault-tolerant extended Kalman filter is a more accurate estimator. The altitude estimation results of FTEKF, shown in magenta dashed line, is much closer to the true altitude shown in black colored dashed-dotted line.

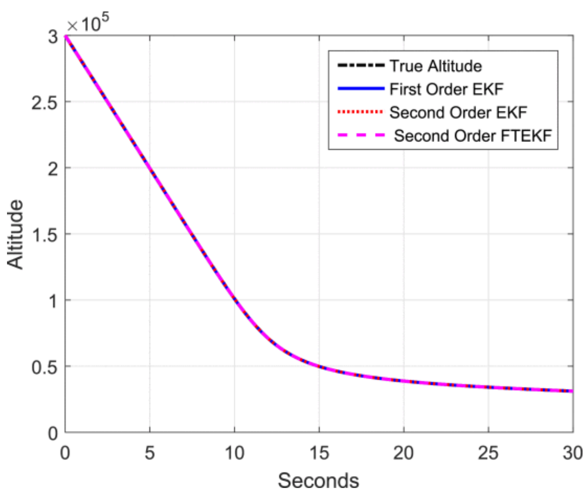

Fig. 2. True altitude and estimated altitude.

The comparisons of true and estimated velocity values using the first-order EKF, the second-order EKF, and the proposed second-order FTEKF are summarized in Fig. 3. Again, under sensor failure conditions, the second-order fault-tolerant extended Kalman filter is a more accurate estimator. The velocity estimation results of FTEKF, shown in magenta dashed line, are much closer to the true velocity shown in black colored dashed-dotted line.

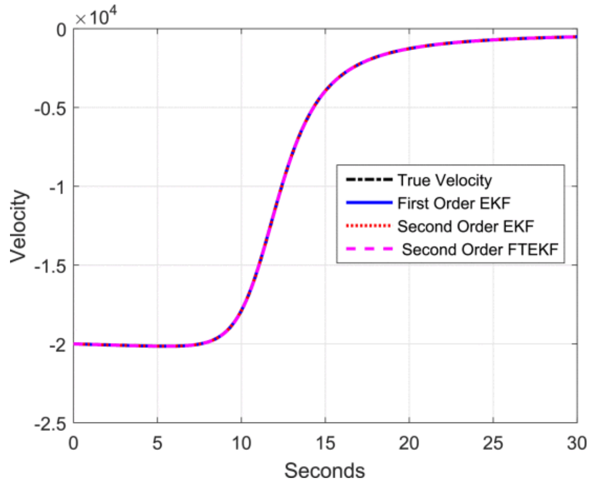

Fig. 3. True velocity and estimated velocity.

To better evaluate the performance improvement of the proposed second-order fault-tolerant extended Kalman filter. The performance metric we used to evaluate second-order FTEKF and other nonlinear estimation methods is the root-mean-square (rms) deviation, which is given as

$$\text{RMS Deviation} = \sqrt{\frac{\sum_{k=1}^{N}(\hat{x}_k - x_k)^2}{N}} \quad (72)$$

where N is the number of time steps.

As shown in Fig. 4, the altitude estimation rms deviation is summarized. The second-order EKF performance is close to the first-order EKF performance, with slightly more accurate estimation results under sensor failure conditions. The first-order EKF rms altitude estimation error is 77.914. The second-order EKF rms altitude estimation error is 77.8874. The second-order FTEKF rms altitude estimation error is 5.3288, which is much more robust against sensor failures.

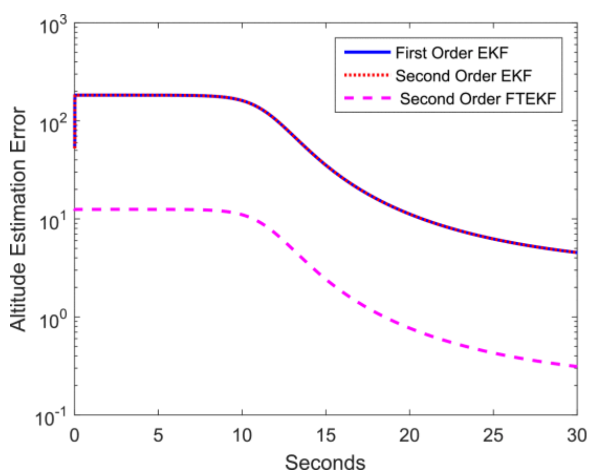

Fig. 4. Altitude estimation RMS comparison.

Similarly, Fig. 5 illustrates the rms deviation comparison of velocity estimation. The second-order EKF performance slightly improves the first-order EKF performance, by including the second-order terms in Taylor series. The first-order EKF rms velocity estimation error is 9.5316. The second-order EKF rms velocity estimation error is 9.5288. The second-order FTEKF rms velocity estimation error is 0.65276, which is a more accurate nonlinear estimator.

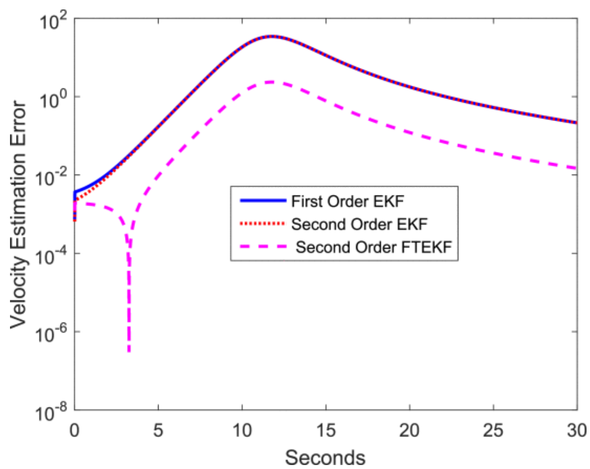

Fig. 5. Velocity estimation rms comparison.

Last but not the least, Fig. 6 illustrates the rms deviation comparison of the constant ballistic coefficient. All nonlinear estimators show good tracking accuracies of the ballistic coefficient. The first-order EKF rms ballistic coefficient estimation error is 9.1315×10^{-3}. The second-order EKF rms ballistic coefficient estimation error is 9.1067×10^{-3}. The second-order FTEKF rms ballistic coefficient estimation error is 9.0002×10^{-3}, which again shows better accuracies.

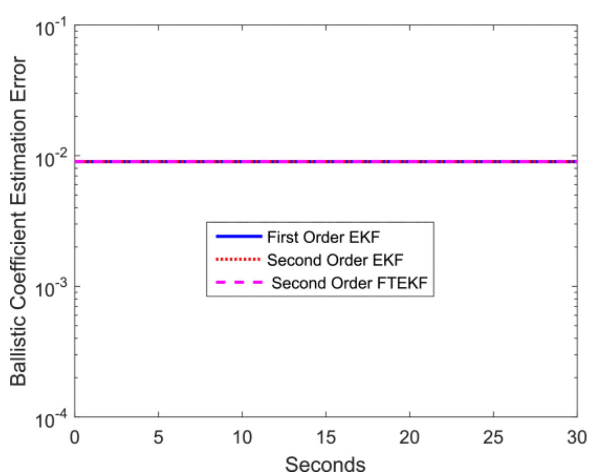

Fig. 6. Constant ballistic coefficient estimation rms comparison.

It should also be noted that various sensor failure rates have been examined. Results are clearly encouraging based on rms deviations of various experiments, we may conclude that the second-order fault-tolerant extended Kalman filter shows superior estimation accuracy, greater robustness, and resiliency in the presence of bad data, external disturbances, and noises.

SECTION VI. Conclusion

Accurate nonlinear estimation under bad data, faulty sensing measurements, extraneous noise, and external disturbances conditions is of great importance in industrial applications. The purpose of this paper is to present a novel second-order fault-tolerant extended Kalman filter. On the basis of a benchmark problem of reconstructing the trajectory of a target with the recorded range measurements from a range-measuring device, computer simulation results indicate that second-order fault-tolerant extended Kalman filter provides superior accuracy than the traditional first- and second-order extended Kalman filter, with similar computational complexity and running time. The proposed second-order fault-tolerant extended Kalman filter is suitable for robust and resilient dynamic state estimation applications.

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