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Abstract

A new approach for characterizing the avalanche-buildup-time-limited bandwidth of avalanche photodiodes (APDs) is introduced which relies on the direct knowledge of the statistics of the random response time. The random response time is the actual duration of the APD's finite buildup-limited random impulse response function. A theory is developed characterizing the probability distribution function (PDF) of the random response time. Recurrence equations are derived and numerically solved to yield the PDF of the random response time. The PDF is then used to compute the mean and the standard deviation of the bandwidth. The dependence of the mean and the standard deviation of the bandwidth on the APD mean gain and the ionization coefficient ratio is investigated. Exact asymptotics of the tail of the PDF of the response time are also developed to aid the computation efficiency. The technique can be readily applied to multiplication models which incorporate dead space and can be extended to cases for which the carrier ionization coefficient is position dependent.

Keywords
Avalanche buildup time, avalanche photodiode, bandwidth, dead space, impact ionization, impulse response, recurrence equations, time response.

1. Introduction

The demand for gigabit-rate fiber communication systems and the rising popularity of fiber optical links in RF/microwave systems have led to a substantial interest in avalanche photodiode (APD) structures with high gain-bandwidth products. However, the gain that APDs provide is accompanied by two types of uncertainty. Specially, randomness in the locations and the times at which impact ionizations take place within the multiplication region of the APD results in fluctuations in the gain and the duration of the time response [1]. The gain noise in APDs have been extensively studied over the years, starting with the pioneering work of McIntyre [2] for a variety of device structures and carrier multiplication models [3-12]. Although gain uncertainty plays an important role in the performance of APDs, it is the finiteness and randomness of the APD time response that give rise to intersymbol interference and can limit the bandwidth of communication systems. The bit error rate, in particular, is known to be dependent on the statistics of the APD random impulse response, or equivalently, on the statistics of the avalanche-buildup-time-limited random bandwidth of the APD.

Traditionally, the statistics of the buildup-time-limited bandwidth of APDs are extracted from the statistics of the APD impulse response, as a function of time following initial photoexcitation. These statistics include the mean and the autocorrelation function of the impulse response function. For example, the mean 3dB bandwidth is directly calculated from the mean frequency response which is simply the Fourier transform of the mean impulse response function. Moreover, fluctuations about the mean 3dB bandwidth can be estimated by considering the one-standard-deviation upper and lower limits of the mean frequency response. Computation of the variance of the frequency response, however, involves taking the Fourier transform of the autocorrelation function of the impulse response. There has been a number of studies on the statistics of the impulse response function for conventional (continuous-multiplication) and staircase APDs of various structures [4, 13-20]. In addition, the statistics of the impulse response function for the dead-space multiplication model, for which the ionizations process has a non-localized nature, have also been studied [21]. Many of the techniques used to calculate the second order statistics of the impulse response in conventional APDs require intensive computations [16,21] and typically involve solving coupled integral or differential equations in four variables. Recently, Bandyopadhyay et al. [22] proposed a computationally simple approach for calculating the mean impulse response function. Their theory, however, does not address the variance and the autocorrelation function of the impulse response.

In this paper, we introduce a new computationally efficient and direct method for calculating the statistics of the buildup-time-limited bandwidth which does not require calculating the statistics of the impulse response function. This method is based on the extracting the statistics of the bandwidth directly from the knowledge of the probability distribution function (PDF) of the random duration of the impulse response, which we will call the random response time. To do so, we develop a recurrence-based theory that characterizes the PDF of the random response time for double-carrier continuous multiplication APDs assuming uniform carrier ionization coefficients in the multiplication region. This model can readily incorporate the effect of dead space which has been receiving attention for the important role it plays in reducing multiplication noise in thin APDs [23-28].The

PDF of the random response time is determined numerically by solving recurrence equations using a simple iteration technique. We also develop an analytical approximation for the tail of the PDF which can further simplify computations. The dependence of the mean and the standard deviation of the bandwidth on the mean gain and the hole-to-electron ionization ratio is then studied. Although the theory assumes uniform ionization coefficients, the recurrence method can be readily extended to devices with strongly position dependent electric fields. The reported theory also has the potential for providing estimates of the statistics of the gain-bandwidth product of APDs [29].

2. Multiplication model

Consider an APD with a multiplication region of width w. A parent photo-electron is injected into the multiplication region at $x = 0$. After traveling a random distance (in the x-direction), called the electron free-path distance, the electron impact ionizes resulting in a secondary electron and a hole. Upon ionization, the regenerated parent electron and the offspring electron continue to travel and may initiate further impact ionizations independently of each other. The offspring hole, on the other hand, travels in the $-x$ −direction and impact ionizes after traveling a hole random free-path distance. The hole ionization results in a secondary hole and an electron. These newly created carriers proceed to generate their own offsprings, and so on. This avalanche of ionization events continues until all carriers exit the multiplication region. In the conventional multiplication theory which is applicable to bulk material (or thick multiplication regions, e.g., in excess of 500 nm), the ability of electrons and holes to effect an impact ionization is not dependent on their past history [2]. This assumption can be translated into a statement about the probability density function (pdf) of the carrier free-path distance. In particular, according to the conventional multiplication theory, the electron and hole free-path distances, respectively denoted by X_e and X_h, are exponentially distributed random variables with pdf's given by

$$f_{X_e}(x) = \alpha e^{-ax}, x \geq 0, (1)$$

and

$$f_{X_h}(x) = \beta e^{-\beta x}, x \geq 0. (2)$$

The parameters α and β are the conventional ionization coefficients for the electron and Hole, respectively. (For example, $f_{X_e}(x)dx$ is the probability that an electron, created at $x = 0$ impact ionizes in the interval $[x, x + dx]$.) Although the conventional theory has been successful in characterizing the multiplication process for a host of experiments with conventional thick-multiplication-region APDs, it cannot predict the change in the gain-noise characteristics in thin APDs[23-26]. A more physical model for carrier impact ionization assumes dependence of the ionization coefficient on the carrier's history. This age-dependent ionization model is often referred to as the dead-space multiplication model [6-8], and it assumes that after each impact ionization a carrier must travel a sufficient distance (the so-called dead space) to gain sufficient energy that will enable it to cause another impact ionization. In the dead-space multiplication model, the pdf's of the carrier free-path distance can be modeled by

$$f_{X_e}(x) = \alpha e^{-\alpha(x-d_e)} u, (x - d_e), (3)$$

and

$$f_{X_h}(x) = \beta e^{-\beta(x-d_h)} u, (x - d_h), (4)$$

where d_e and d_h are the electron and hole dead spaces, respectively, and $u(x)$ is the unit step function (i.e., $u(x) = 1$ if $x \geq 0$ and zero otherwise). It is important to note that the ionization coefficients in the dead-space model are different from the coefficients appearing in (1) and (2) [28]: These nonlocalized coefficients correspond to *enabled* carriers that have traveled the dead space.[1] In the theory of the statistics of the response time, the form of the pdf of the carrier free-path distance is arbitrary. Our approach is therefore applicable to both the conventional and dead-space ionization models. Furthermore, more realistic age-dependent ionization models for which the newly created carriers gradually attain ionization ability can also be incorporated by simply modifying the forms of the pdf's of the free-path distance. The specific forms are typically obtained by means of Monte-Carlo simulation [26].

3. Statistics of The Random Response Time and Random Bandwidth

We begin this section by defining the random response time, T, as the time measured from the instant a parent electron enters the multiplication region (at $x = 0$) to the time when all carriers exit the multiplication region. T is merely the duration of the random impulse response function $I(t)$ of the APD. We adopt the traditional definition of bandwidth and define the buildup-limited bandwidth of the APD as

$$B = 1/T. \quad (5)$$

In this section, we characterize and compute the PDF $F_T(t) = \mathrm{P}\{T \leq t\}$ and use it to determine the statistics of the bandwidth. The mean and the variance of the random bandwidth can be readily calculated using

$$\langle B \rangle = \int_0^\infty t^{-1} f_T(t) dt, \quad (6)$$

and

$$\sigma_B^2 = \int_0^\infty (t^{-1} - \langle B \rangle)^2 f_T(t) d\,t, \quad (7)$$

where $f_T(t) = dF_T(t)/dt$ is the probability density function of T.

3.1 Recurrence Equations

The characterization of $F_T(t)$ is done by means of deriving a pair of coupled recurrence equations. To formulate the recurrence equations, we need to define certain intermediate random response times. Let $T_e(x)$ denote the duration of the random impulse response function when the multiplication is initiated by a single parent *electron positioned at x* in the multiplication region. Similarly, let $T_h(x)$ be the duration of the random impulse response function when the multiplication is initiated by a single parent *hole positioned at x*. Note that $T_e(0)$ is simply the random response time T.

The rationale behind our recurrence approach is the fundamental fact that once a parent carrier impact ionizes, the regenerated parent carrier and the offspring carriers independently repeat a similar process as their parent. For example, suppose that the avalanche process is initiated by a single parent electron at position x and that the first impact ionization of this parent electron occurs at position $x + \xi$ (i.e, the electron random free-path distance X_e assumes the value ξ. Then, conditional on the location of the first impact ionization being $x + \xi$, the event $\{T_e(x) \leq t\}$ occurs if and only if the responses corresponding to all three newly created carriers at position $x + \xi$ terminate in the remaining time. This remaining time is precisely t less the transit time of the electron from x to $x + \xi$. Since we assume that the two electrons and the hole act statistically independently of one another, the conditional probability of the occurrence of the event $\{T_e(x) \leq t\}$ given that the first ionization occurs at $x + \xi$ is the product of the probabilities that the responses due to the three carriers all terminate before $t - \xi/v_e$ where v_e is the electron saturation (or drift) velocity in the multiplication region. (As in other models reported in the literature [18], we use the carrier drift velocity as an approximation avoiding the need for accounting for the stochastic nature of the carrier instantaneous velocity.) A similar argument can be developed if a parent hole initiates the multiplication process.

Keeping the above concept of regeneration of events in mind, we now derive two coupled recurrence equations characterizing the probability distribution functions of $T_h(x)$ and $T_e(x)$. Let $F_e(t,x) = P\{T_e(x) \le t\}$ and $F_h(t,x) = \mathrm{P}\{T_h(x) \le t\}$ be the PDF's $T_e(x)$ of and $T_h(x)$, respectively. Clearly, $F_T(t) = F_e(t,0)$, since $T_e(0) = T$. First note that the probability that the parent electron (positioned at x) never ionizing is $1 - \int_0^{w-x} f_{X_e}(\xi)d\xi$. In this case, T is precisely $(w-x)/v_e$, and hence, $F_e(t,x) = 0$ if $t < (w-x)/v_e$ and $F_e(t,x) = 1$ if $t \ge (w-x)/v_e$. Now suppose that the first ionization takes place at position $x + \xi$, where the free-path distance ξ is in the interval $[0, w-x]$. According to the preceding regeneration concept, the conditional probability of the occurrence of the event $\{T_e(x) \le t\}$ given that the first ionization occurs at $x + \xi$, denoted by $\mathrm{P}\{T_e(x) \le t | X_e = \xi\}$, is given by

$$\mathrm{P}\{T_e(x) \le t | X_e = \xi\} = \mathrm{P}\{T_e(x+\xi) \le t - \xi/v_e\}^2 \cdot \mathrm{P}\{T_h(x+\xi) \le t - \xi/v_e\}. \quad (8)$$

Now by averaging the above conditional probability over all possible locations where the first ionization may have occurred (i.e. ξ in the range $0 \le \xi \le w - x$), we obtain $F_e(t,x)$. Hence, for $t \ge (w-x)/v_e$, we obtain the recurrence equation

$$F_e(t,x) = 1 - \int_0^{w-x} f_{X_e}(\xi)d\xi + \int_0^{w-x} F_e^2(t - \xi/v_e, x+\xi) \cdot F_h(t - \xi/v_e, x + \xi) f_{X_e}(\xi)d\xi. \quad (9)$$

Similarly, we can start the avalanche multiplication with a single parent hole at x and follow a similar conditioning argument to obtain a recurrence equation for $F_h(t,x)$. In particular, for $t < x/v_h$, $F_h(t,x) = 0$, and for $t \ge x/v_h$,

$$F_h(t,x) = 1 - \int_0^{x} f_{X_h}(\xi)d\xi + \int_0^{x} F_h^2(t - \xi/v_h, x+\xi) \cdot F_e(t - \xi/v_h, x + \xi) f_{X_h}(\xi)d\xi. \quad (10)$$

Where v_h is the drift velocity of the hole in the multiplication region.

The above recurrence equations can be cast into a standard form using normalized time and space variables. In particular, by replacing x and t by the normalized variables x/w and t/τ_e, respectively, where τ_e is the electron transit time across the multiplication region, we can recast (9) and (10): For $t < 1 - x$, $F_e(t,x) = 0$, and for $t \ge 1 - x$,

$$F_e(t,x) = 1 - \int_0^{1-x} \tilde{f}_{X_e}(\xi)d\xi + \int_0^{1-x} F_e^2(t - \xi, x+\xi) \cdot F_h(t - \xi, x + \xi) \tilde{f}_{X_e}(\xi)d\xi. \quad (11)$$

Similarly, for $t < x\, v_e/v_h$, $F_h(t,x) = 0$, and for $t \ge x\, v_e/v_h$

$$F_h(t,x) = 1 - \int_0^{x} \tilde{f}_{X_h}(\xi)d\xi + \int_0^{x} F_h^2(t - \xi v_e/v_h, x+\xi) \cdot F_e(t - \xi v_e/v_h, x + \xi) \tilde{f}_{X_h}(\xi)d\xi. \quad (12)$$

Note that the new dimensionless variable x takes values in the interval $[0,1]$ and the scaled density functions of the carrier free path are given by

$$\tilde{f}_{X_e}(x) = wf_{X_e}(wx) \quad (13)$$

and

$$\tilde{f}_{X_h}(x) = wf_{X_h}(wx). \quad (14)$$

The device parameters in these equations are now all conveniently normalized and they are: the electron and hole ionization parameters αw and βw, respectively, the relative electron and hole dead spaces d_e/w and d_h/w, respectively, and the ratio of the electron to hole drift velocities v_e/v_h.

3.2 Numerical Solution of the Recurrence Equations

Equations (11) and (12) can be solved numerically using the simple iterative method described below. We first select a maximum limit R for the range of the normalized time to be consid- ered. We then select a mesh size for the normalized time and the normalized space x allowing the discretization of the functions F_e and F_h, and hence, converting the integrals into sum- mations. We set the zeroth iteration of the functions $F_e(t,x)$ and $F_h(t,x)$, denoted, respectively, by $F_{e0}(t,x)$ and $F_{h0}(t,x)$, to zero for $0 \le x \le 1$ and $0 \le t \le R$. The first iterates, $F_{e1}(t,x)$ and $F_{h1}(t,x)$, are computed by substituting the zeroth iterates into the right-hand side of (11) and (12). Similarly, the second iterates, $F_{e2}(t,x)$ and $F_{h2}(t,x)$, are, respectively, computed from the first iterates by substituting $F_{e1}(t,x)$ and $F_{h1}(t,x)$, into the right-hand side of (11) and (12) and carrying out the summations. Subsequent iterates are generated in the same way. This procedure is continued until the maximum, over the range of values of x and t, of the relative change in the functions $F_{en}(t,x)$ and $F_{hn}(t,x)$, is below a predefined tolerance level. A number of specific examples of this procedure are considered in Section IV.

The above numerical technique can become computationally intensive if the range R is large (e.g., in excess of 100). Accurately calculating the statistics of T and B, however, may require knowledge of $F_T(t)$ far beyond $t = 100$ in cases when the response time is large (e.g., in GaInAs where the ionization ratio is nearly 0.5). Fortunately, the asymptotic behavior of $F_T(t)$ can be analytically represented by an exponential model which can be extracted from (11) and (12), thus avoiding the need for considering large values of R.

3.3 Analytical Approximation of the Tail of the Probability Distribution Function of the Random Response Time

For brevity, we will consider the case when dead space is ignored. We also make the simplifying assumption that the electron and hole drift velocities are equal. The more general cases which may include dead space and/or more physical assumption on the drift velocities can be addressed in a similar way. We begin by proposing that for sufficiently large t,

$$F_e(t,\xi) \approx 1 - A(\xi)e^{-\gamma t} \quad (15)$$

and

$$F_h(t,\xi) \approx 1 - B(\xi)e^{-\gamma t} \quad (16)$$

where $A(\xi)$ and $B(\xi)$ are spatial functions to be determined later, and γ is an exponential temporal rate. Note that γ is also the exponential decay rate of the pdf $f_T(t)$. It is shown in the Appendix that the decay rate is the solution to the nonlinear equation

$$(r_2(\gamma) + \alpha + \gamma)e^{r_1(\gamma)} + (r_1(\gamma) + \alpha + \gamma)e^{r_2(\gamma)} = 0 \quad (17)$$

Where $r_1(\gamma)$ and $r_2(\gamma)$ are the roots of the equation

$$r^2 + r(\alpha + \beta) - \gamma(\alpha + \beta + \gamma) = 0. \quad (18)$$

The solution to the nonlinear equation (17) can be easily generated by means of standard numerical recipes. In Section IV, we use the MATLAB function *fzeros* to solve (17). In practice, for a given set of device parameters, the tail of the density function f_T can be estimated as follows: First, the exponential decay is determined using (17) and (18). Then, the model $A(0)e^{-\gamma t}$ is fitted to the tail of f_T and $A(0)$ is determined.

4. Discussion

In this section we apply the theory of Section III to characterize the behavior of the mean and standard deviation of B. For simplicity, we will assume that holes and electrons have equal velocities in the multiplication region. Furthermore, we will present the results considering only the conventional mul- tiplication model for which the carrier free path densities are given by (1) and (2). The recurrence equations (11) and (12) are solved numerically using various device parameters and the probability distribution function $F_T(t)$ is computed. Fractional time and distance of magnitudes 10^{-3} and 5×10^{-3}, respectively, are used to define the mesh size for the recurrence equations (11) and (12). Three cases of the hole-to-electron ionization ratio k are considered ($k = 0.1, 0.5$, and 1.0) representing a range of devices with almost single-carrier multiplication to strongly double-carrier-multiplication characteristics. For each case of k, the electron ionization parameter αw is selected so

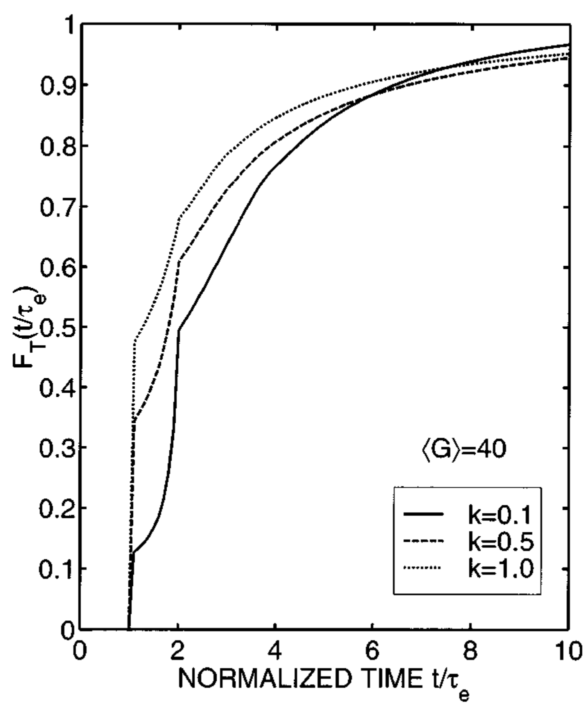

Fig. 1. Probability distribution function of the random response time as a function of the normalized time t/τ_e, where τ_e is the electron transit time across the multiplication region. The APD mean gain is 40. Three values for the hole-to-electron ionization coefficient ratio k are considered: $k = 0.1, 0.5$, and 1.0.

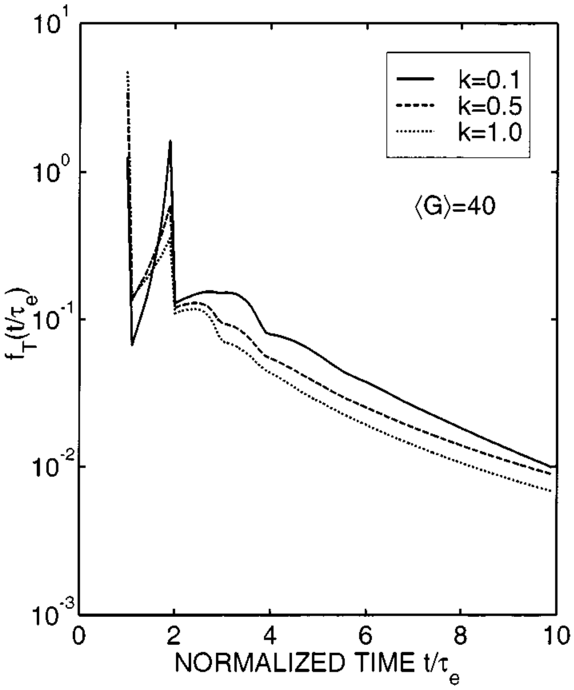

Fig. 2. Probability density function of the random response time as a function of the normalized time t/τ_e, where τ_e is the electron transit time across the multiplication region for the same device parameters used in Fig. 1. Note the exponential decay of the tail.

that the mean gain is 40, according to the well-known McIntyre formula for the APD gain [1], [2]. Fig. 1 depicts the behavior of F_T as a function of the normalized time t/τ_e. Each curve required approximately 25 min to generate using a 66 MHz SUN workstation. The probability density function f_T is obtained from F_T by simply taking the derivative (numerically), as shown in Fig. 2. The predicted exponential tail (e.g., when t/τ_e is approximately in excess of five in the examples considered) is clear. The impulse in f_T (or jump in F_T) at $t/\tau_e = 1$ is due to the possibility that the initial parent carrier undergoes no impact ionizations, in which case T is exactly τ_e. The impulse area (or jump size in F_T) is therefore precisely the probability that the parent electron does not impact ionize and it is simply

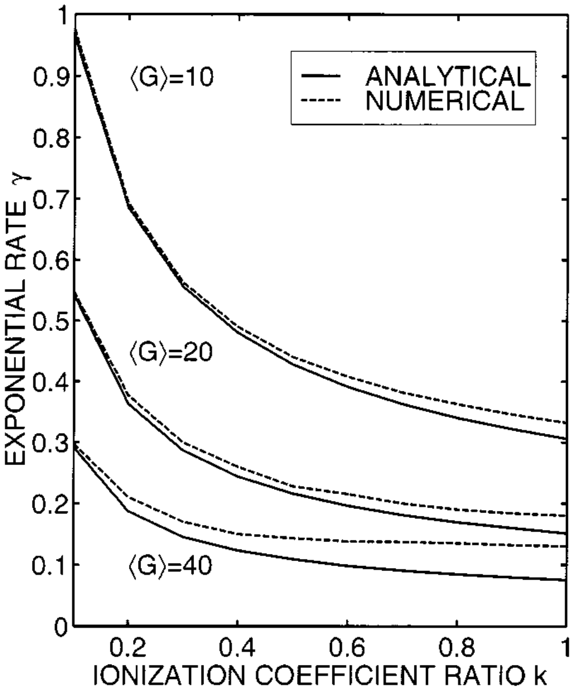

Fig. 3. A comparison between the predicted analytical exponential decay rate and the numerically calculated decay rate using the recurrence equation. Three values for the APD mean gain are considered: $\langle G \rangle = 10,\ 20$, and 40

equal to $e^{-\alpha w}$. The fact that f_T decays to zero (as t increases) more slowly as k increases is indicative of the longer buildup time associated with the increase in k.

By computing and plotting the mean and the variance of T as a function of the ionization coefficient ratio k, it has been observed that both quantities increase with the increase in k. This behavior is expected from the physical fact that for two electron-initiated devices (fabricated from different materials) with the same mean gain and multiplication-region widths, more holes impact ionize (on average) in the device with the higher k than that of the lower k. As a result, a greater portion of the total ionizations are due to secondary impact ionizations leading to a longer response time. The analytical approximation of the exponential decay rate of $f_T(t)$ (see Section III-C) were particularly valuable in the calculations since it allowed breaking up the integrals corresponding to $\langle T \rangle$ and $\langle T^2 \rangle$ into two parts: one part extending over the range of values of T below the maximum normalized time , and the other part extending beyond R and involving the predicted exponential expression for f_T.

In order to verify the validity of the analytical asymptotic exponential expression for f_T, we compared the predicted exponential decay rate obtained using (17) with the decay rate calculated from Fig. 2. Fig. 3 shows the predicted and the numerically computed rates as functions of the ionization coefficient ratio k with the APD mean gain $\langle G \rangle$ used as a parameter. The agreement between the two rates is generally good, especially when the mean gain is low to moderate (i.e., $\langle G \rangle = 10$ and 20). This error, however, increases as the the mean gain increases (e.g., $\langle G \rangle = 40$) or when k is high (i.e., in excess of 0.5). The reason for this behavior for such cases of high gain is that since the avalanche buildup time is high, a longer range of time in the numerical calculations of the recurrence equations is needed before the analytical asymptotic rates are achieved.

We now turn to computing the mean and the standard deviation of B. Observe that due to the normalization of T by the electron transit time τ_e, the bandwidth is also normalized: The normalized bandwidth B is the fraction of the electron-transit-time-limited bandwidth, $B_{\max}$, defined by $B_{\max} = \tau_e^{-1}$. Note that $B_{\max}$ is the absolute maximum bandwidth possible and it is achieved only when no impact ionizations take place. The dependence of mean bandwidth $\langle B\rangle$ on the APD mean gain $\langle G\rangle$ is shown in Fig. 4. As expected, the reduction in the bandwidth is evident as the mean gain increases. We have observed, however, that for a fixed gain, an increase in k does not necessarily yield a higher mean bandwidth. This result seems, at a first glance, to be at odds with the fact that the mean response time $\langle T\rangle$ increases with the increase in k. This apparent paradox can be explained as follows. For a fixed gain, an increase in k must be accompanied by a reduction in the product αw, which in turn, increases the likelihood of the event that the parent electron never impact ionizes (recall that the probability of this event is $e^{-\alpha w}$). When such an event occurs, the resulting gain is unity and the corresponding response time is equal to the electron transit time across the multiplication region. It turns out that as far as the computation of the mean bandwidth is concerned, the effect of increased secondary ionizations (which is due to an increase in k) is overcompensated by the increase in the likelihood of the short response time corresponding to unity gain realizations. It is important to note, however, that the effect of this increase in the mean bandwidth is not expected to be very significant on the statistics of the gain-bandwidth product since such realizations with high bandwidth are accompanied with a relatively low gain value.

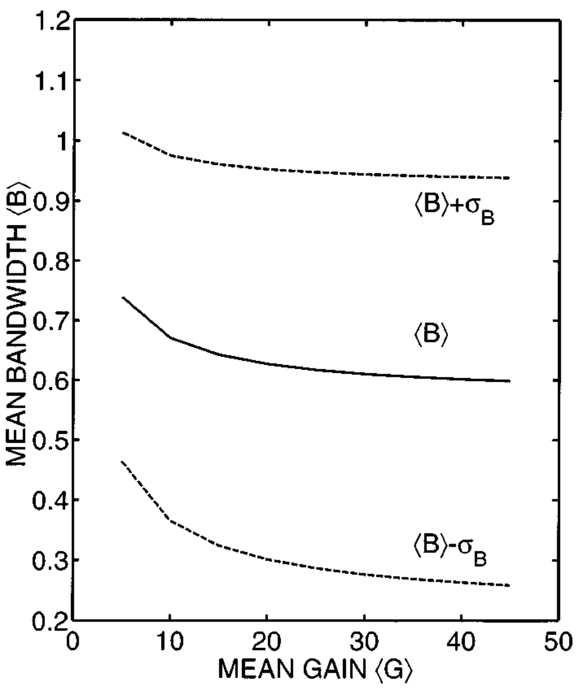

Fig. 4. Dependence of the mean bandwidth on the mean gain for a device with an ionization coefficient ratio of $k = 0\ 5$. Dashed lines represent the single standard deviation upper and lower limits.

Fluctuation in the bandwidth relative to the mean bandwidth is shown in Fig. 5, where the ratio between the standard deviation of the bandwidth and the mean bandwidth, $\sigma_B/\langle B\rangle$, is plotted as a function of the mean gain $\langle G\rangle$. It is seen that relative uncertainty in the bandwidth increases with the mean gain. This interesting behavior resembles, in essence, the gain-noise characteristics of APD's where the excess noise factor increases with the increase in the mean gain. Here too, we see that the un- certainty (or noise) in the bandwidth, is dependent on the mean gain in a similar way. We expect this uncertainty in the band- width to play an important effect on the bit-error rates of high data-rate communication systems in a fashion similar to the effect that the excess noise factor has on the system performance.

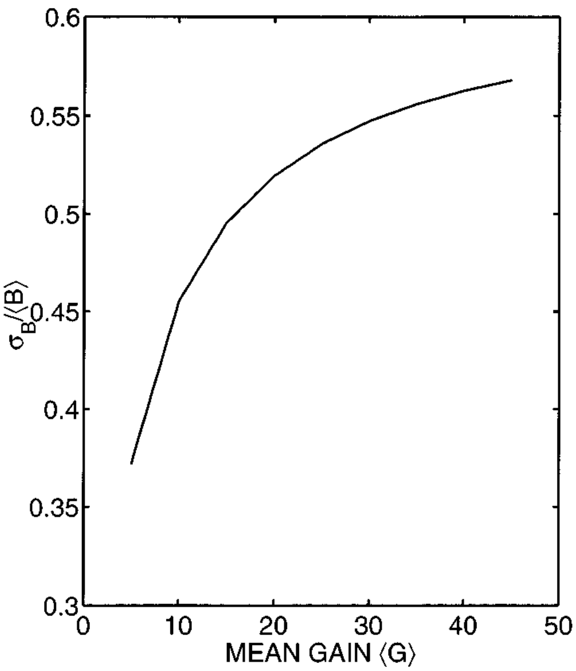

Fig. 5. Ratio of the standard deviation of the bandwidth to the mean bandwidth as a function of the mean gain for a device with an ionization coefficient ratio of $k = 0.5$. Uncertainty in the bandwidth increases with the increase in the APD mean gain.

5. Conclusions

We developed a new technique that facilitates computing the statistics of the avalanche-buildup-time-limited bandwidth of APD's. The method is based on certain recurrence equations that characterize the probability distribution function (PDF) of the random avalanche buildup time. Since uncertainty in the bandwidth affects the bit-error rate in high data-rate communication systems in a fashion similar to the way APD gain un- certainty degrades the performance, it is important to have a model that can estimate these bandwidth fluctuations. The standard deviation of the bandwidth relative to the mean bandwidth is seen to increase with the increase in the mean gain. This behavior resembles the dependence of the uncertainty in the APD gain (represented by the excess noise factor) on the mean gain. In comparison to the traditional methods for determining the mean and the standard deviation of the bandwidth, the reported technique is very efficient since the only significant computation is the calculation of the PDF of the random response time. This calculation is computationally equivalent to calculating the mean impulse response of an APD. However, once the probability distribution is calculated, any statistic, including the mean and standard deviation, of the random bandwidth can be calculated with minimal effort. In contrast to the reported technique, estimating the standard deviation of the bandwidth from the statistics of the impulse response function, as traditionally done, requires knowledge of autocorrelation function of the impulse response, a task that is computationally intensive. In fact, knowledge of the statistics of the random response time can be used to efficiently estimate the mean, the variance, and the auto-correlation function of the impulse response function. For example, a simple model for the random impulse response can be taken as a rectangular or triangular pulse of a duration equal to the random response time and with a total area equal to the APD random gain. Finally, the reported technique is readily applicable to the dead-space multiplication model. This gives the theory the potential to be used in determining the effect of dead space on the bandwidth in thin APD's which have recently received considerable attention due to their low multiplication-noise characteristics.

APPENDIX

ANALYTICAL APPROXIMATION OF THE TAIL OF $F_t(t)$

We begin by substituting the forms (15) and (16) into (11) and (12), multiplying both sides by $e^{\gamma t}$, and taking the limit as $t \to \infty$ to obtain two integral equations involving $A(\xi)$, $B(\xi)$, and the exponential rate γ:

$$A(\xi) = \int_0^{1-\xi} \alpha(2\alpha A(\xi + s) + B(\xi + s))e^{(\gamma-\alpha)s}ds \quad (19)$$

and

$$B(\xi) = \int_0^{x} \beta(2B(\xi - s) + A(\xi - s))e^{(\gamma-\beta)s}ds. \quad (20)$$

The integral equations (19) and (20) can be easily converted to the following differential equations [for example, use a change of variable $u = \xi + s$ in (19) and then differentiate with respect to ξ]:

$$A'(\xi) = -\alpha\big(A(\xi) + B(\xi)\big) - \gamma A(\xi) \quad (21)$$

and

$$B'(\xi) = -\beta\big(B(\xi) + A(\xi)\big) - \gamma B(\xi). \quad (22)$$

The above linear equations assume a solutions of the form $A(\xi) = c_1 e^{r\xi}$ and $B(\xi) = d_1 e^{r\xi}$. We can calculate the exponent by substituting the above forms into (21) and (22). system of self-consistency equations given by

$$\begin{bmatrix} r+\alpha+\gamma & \alpha \\ -\beta & r-\beta-\gamma \end{bmatrix}\begin{bmatrix} c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (23)$$

The above system of linear homogeneous equations has a non- trivial solution if and only if the matrix in (23) is singular. By setting the determinant of this matrix to zero, we obtain a non- linear characteristic equation whose two zeros are values of the exponent r which yield a solution to (21) and (22). This char- acteristic equation is given by (18). We now form the general solution to (21) and (22), respectively, as

$$A(\xi) = c_1 e^{r_1\xi} + c_2 e^{r_2\xi} \quad (24)$$

and

$$B(\xi) = d_1 e^{r_1\xi} + d_2 e^{r_2\xi} \quad (25)$$

The constants c_2 and d_2 can be related to c_1 and d_1, respectively, through the boundary conditions for $A(\xi)$ and $B(\xi)$. In particular, using the fact that the duration of a response generated as a result of a parent electron at location $x = 1$ is zero [i.e., $T_e(1) = 0$], it is clear that $F_e(t.1) = 1$ for all t. Similarly, since $T_h(0) = 0$, $F_h(t, 0) = 1$. Using these facts in (15) and (16) yields the boundary conditions $A(1) = 0$ and $B(0) = 0$. Now by applying these boundary conditions in (24) and (25), we obtain

$$c_1 e^{r_1} + c_2 e^{r_2} = 0 \quad (26)$$

and

$$d_1 + d_2 = 0. \quad (27)$$

Observe that for every $\gamma > 0$, we can find the zeros of (18) and obtain the exponents $r_1(\gamma)$ and $r_2(\gamma)$. However, there is only one value of γ for which the boundary conditions (26) and (27) are satisfied. To find this special γ, note that if we substitute (24) and (25) into (21) we obtain

$$(r_2 + \alpha + \gamma)c_2 + (r_1 + \alpha + \gamma)c_1 = 0. \quad (28)$$

Finally, by combining (28) with (26) and (27) we arrive at the nonlinear equation given in (17).

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