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Paul Bankston
Marquette University, paul.bankston@marquette.edu

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On continuous images of ultra-arcs

Paul Bankston
Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI

Abstract

Any space homeomorphic to one of the standard subcontinua of the Stone-Čech remainder of the real half-line is called an *ultra-arc*. Alternatively, an ultra-arc may be viewed as an ultracopower of the real unit interval via a free ultrafilter on a countable set. It is known that any continuum of weight $\leq \aleph_1$ is a continuous image of any ultra-arc; in this paper we address the problem of which continua are continuous images under special maps. Here are some of the results we present.

- •Every nondegenerate locally connected chainable continuum of weight $\leq \aleph_1$ is a co-elementary monotone image of any ultra-arc.
- •Every nondegenerate chainable metric continuum is a co-existential image of any ultra-arc.
- •Every chainable continuum of weight $\aleph_1$ is a co-existential image of any ultra-arc whose indexing ultrafilter is a Fubini product of two free ultrafilters.
- •There is a family of continuum-many topologically distinct nonchainable metric continua, each of which is a co-existential image of any ultra-arc.
- •A nondegenerate continuum which is either a monotone or a co-existential image of an ultra-arc cannot be aposyndetic–let alone locally connected–without being a generalized arc.

Keywords

Continuum, Metric continuum, (Generalized) arc, Ultra-arc, Co-elementary map, Co-existential map, Monotone map, Locally connected, Aposyndetic, Antisymmetric, (Hereditarily) decomposable, Co-existentially closed, Chainable, Span, Stone-Čech remainder

1. Introduction

Ultra-arcs are (homeomorphs of) the so-called *standard subcontinua* of the Stone-Čech remainder $\mathbb{H}^*:=\beta(\mathbb{H})\setminus\mathbb{H}$ of the real half-line $\mathbb{H}:=[0,\infty)$. Ultra-arcs resemble arcs in many–but far from all–ways; standard subcontinua are widely regarded as building blocks for $\mathbb{H}^*$, in rough analogy with how arcs are building blocks for solenoids. In this paper we look into the problem of identifying continua which can be images of ultra-arcs under special maps.

Continuing the analogy with arcs, we know that it is the continua that are locally connected and metric which can be continuous images of an arc, but only another arc can be a monotone–or even a confluent–image. When we move to ultra-arcs, *every* continuum of weight $\leq \aleph_1$ is a continuous image of any ultra-arc, but nondegenerate monotone images have to be decomposable (among other things).

Let us begin with some basic definitions. A **compactum** is a compact Hausdorff space; a **continuum** is a connected compactum, and a **subcontinuum** of a topological space is a subset whose induced topology makes it a continuum. A space is **nondegenerate** if it has at least two points. A **cut point** of a continuum is any point whose complement is disconnected; it is well known [31] that every nondegenerate continuum has at least two noncut points. One with exactly two is called a **generalized arc**. An **arc** is a generalized arc which is metric; all arcs are homeomorphic [26] to the closed unit interval $\mathbb{I}:=[0,1]$ in the real line. Generalized arcs are also characterized as being those nondegenerate continua whose topologies are induced by a linear order on the underlying set. For this reason we refer to a continuum as being **linear** if it is either degenerate or a generalized arc. (Hence the nonlinear continua are the ones with at least three noncut points.)

A continuum X is **irreducible about** a subset *S* (or, *S* is a **set of irreducibility** for X) if no proper subcontinuum of X contains *S*. All continua are irreducible about their sets of noncut points [31]. X is **irreducible** if it is irreducible about a doubleton subset. If X is irreducible about {x,y} and every closed set that is a set of irreducibility for X contains $\{x,y\}$, then we say that X is **uniquely irreducible**. When this happens, any autohomeomorphism on X fixes $\{x,y\}$ setwise, and it makes sense to refer to x and y as the **end points** of X. When X is a generalized arc it is uniquely irreducible, and its end points are the order end points for any linear order inducing its topology.

A **standard subcontinuum** of $\mathbb{H}^*$ is a subset of the form

$$\bigcap_{J\in\mathcal{D}} \mathrm{cl}_{\beta(\mathbb{H})}\left(\bigcup_{n\in J}[a_n,b_n]\right),$$

where $a_0<b_0<a_1<b_1<\cdots$ is an increasing unbounded sequence in $\mathbb{H}$ and $\mathcal{D}$ is a free (i.e., nonprincipal) ultrafilter on $\omega:=\{0,1,2,\ldots\}$. This set is easily shown to be a subcontinuum of $\mathbb{H}^*$; $\mathcal{D}$ is its **indexing ultrafilter**. Two standard subcontinua with the same indexing ultrafilter are easily seen to be homeomorphic; an **ultra-arc** is any continuum homeomorphic to a standard subcontinuum of $\mathbb{H}^*$.

An alternative description of ultra-arcs, using just the single interval $\mathbb{I}$, starts with forming the Stone-Čech compactification $\beta(\mathbb{I} \times \omega)$. Letting $q: \mathbb{I} \times \omega \to \omega$ denote projection onto the second coordinate, we now consider the natural lift $q^\beta: \beta(\mathbb{I} \times \omega) \to \beta(\omega)$. Points of the image space are just the ultrafilters on ω, and we define $\mathbb{I}_\mathcal{D}$ to be the pre-image of $\mathcal{D} \in \beta(\omega)$ under q^β. If $\mathcal{D}$ is the principal ultrafilter fixed at $n \in \omega$, we obtain $\mathbb{I}_\mathcal{D} = \mathbb{I} \times \{n\}$. On the other hand, if $\mathcal{D} \in \omega^*$, $\mathbb{I}_\mathcal{D}$ is homeomorphic to any ultra-arc indexed by $\mathcal{D}$. The subspaces $\mathbb{I}_\mathcal{D}$, $\mathcal{D} \in \beta(\omega)$, constitute the components of $\beta(\mathbb{I} \times \omega)$. The introduction of ultra-arcs to study $\mathbb{H}^*$ is due to J. Mioduszewski [24], and a thorough summary of their use in this connection may be found in [18].

Our main interest is the general problem of determining when a continuum is an image of an ultra-arc under continuous maps satisfying extra conditions. With no conditions on the maps, there is an elegant solution.

Theorem 1.1

(i) ([28, Theorem 9.20], attributed to D. Bellamy) *If X is any metric continuum and A is a nondegenerate subcontinuum of $\mathbb{H}^*$, then there is a continuous map from A onto X.*

(ii) ([16, Theorem 1, also Section 4]) *If X is any continuum of weight $\leq \aleph_1$ and A is either $\mathbb{H}^*$ or one of its standard subcontinua, then there is a continuous map from A onto X.*

Here we consider the mapping conditions of being monotone and of being co-existential/co-elementary. While monotone maps are well studied in continuum theory, co-existential and co-elementary maps are relatively new, arising as natural category-theoretic duals to existential and elementary embeddings, respectively, in model theory (see, e.g., [7]). Co-elementary maps are co-existential, but not conversely; co-existential maps with locally connected range are monotone.

The results of this paper are summarized as follows: (1) Every nonlinear monotone image of an ultra-arc contains a nondegenerate indecomposable subcontinuum. (2) Every nondegenerate monotone image of an ultra-arc is decomposable. (3) Every generalized arc–indeed, every nondegenerate locally connected chainable continuum–of weight $\leq \aleph_1$ is a co-elementary monotone image of any ultra-arc. (4) Every nondegenerate chainable metric continuum is a co-existential image of any ultra-arc. (5) Every chainable continuum of weight $\aleph_1$ is a co-existential image of any ultra-arc whose indexing ultrafilter is a Fubini product of two free ultrafilters. (6) There is a family of continuum-many topologically distinct metric continua which are nonchainable–indeed, of nonzero span–and are co-existential images of any ultra-arc. (7) Every aposyndetic monotone (or co-existential) image of an ultra-arc is linear.

2. Monotone maps and co-existential maps

A map from one topological space to another is **monotone** if its point pre-images are connected; it is well known that continuous $f: X \to Y$ between compacta is monotone if and only if the pre-image $f^{-1}[K]$ of a subcontinuum $K \subseteq Y$ is a subcontinuum of X. Monotone maps are closely related to what we call *co-existential* maps, but to define the latter notion we need to expand on the "alternative" description of ultra-arcs given in the Introduction.

The topological ultracopower–more generally, ultracoproduct–construction for compacta was initiated in [2]; also, independently (in the case of arcs), by Mioduszewski [24]. We start with a compactum X and an infinite set I, viewed as a discrete topological space. With $p: X \times I \to X$ and $q: X \times I \to I$ the coordinate projection maps, we apply the Stone-Čech functor to obtain the diagram

$$X = \beta(X) \overset{p^\beta}{\longleftarrow} \beta(X \times I) \overset{q^\beta}{\longrightarrow} \beta(I).$$

Regarding an ultrafilter $\mathcal{D}$ on the set I as a point in $\beta(I)$, we form the D-**ultracopower** $X_\mathcal{D}$ as the pre-image $(q^\beta)^{-1}[\mathcal{D}] := (q^\beta)^{-1}[\{\mathcal{D}\}]$. (This is precisely the description of ultra-arcs when $X = \mathbb{I}$ and $\mathcal{D} \in \omega^*$.) It is a basic fact [3], [7] about this construction that $X_\mathcal{D}$ is a continuum if and only if X is a continuum. In this case the family $\{X_\mathcal{D}: \mathcal{D} \in \beta(I)\}$ comprises the components of $\beta(X \times I)$.

The restriction $p_{X,\mathcal{D}} := p^\beta | X_\mathcal{D}$, called the **codiagonal map**, is a continuous surjection from $X_\mathcal{D}$ to X. A continuous surjection $f: X \to Y$ between compacta is **co-existential** if there is an ultrafilter $\mathcal{D}$ and a continuous surjection $g: Y_\mathcal{D} \to X$ such that $f \circ g = p_{Y,\mathcal{D}}$. If g can be chosen to be of the form $p_{X,\mathcal{E}} \circ h$, where $h: Y_\mathcal{D} \to X_\mathcal{E}$ is a homeomorphism, we call f a **co-elementary** map. (These notions are exact category-theoretic duals to those of *existential embedding* and *elementary embedding*, respectively, in model theory, and do not explicitly mention topological properties of subsets of either the domain or the range.) Because ultracopowers of degenerate continua are degenerate, it is immediate from the definition that a co-existential image of a nondegenerate continuum is nondegenerate.

A topological space is **locally connected** if there is an open base for the space consisting of connected sets. The following is a very useful fact for us.

Lemma 2.1
[6, Theorem 2.7] Co-existential maps with locally connected range are monotone.

Remark 2.2
Although we do not use the fact here, it is also true (see, e.g., [12, Lemma 3.14]) that if X is a compactum which is not locally connected, then there is a codiagonal map $p_{X,\mathcal{D}}$ which is not monotone.

3. Regular closed sets
If X is a compactum, then one way of viewing the ultracopower $X_\mathcal{D}$ is as the Wallman space of maximal filters in the lattice of all $\mathcal{D}$-ultraproducts $\vec{F}^\mathcal{D} := \prod_\mathcal{D} F_i$ of I-sequences of sets closed in X. $\vec{F}_\mathcal{D}$, also denoted $\sum_\mathcal{D} F_i$ and defined to be $\{\mu \in X_\mathcal{D}: \vec{F}^\mathcal{D} \in \mu\}$, is a basic closed subset of $X_\mathcal{D}$. Closed subsets of this form are called **regular**. If each F_i is a singleton $\{x_i\}$ then $\vec{F}_\mathcal{D}$ has just one point, which we denote $\vec{x}_\mathcal{D}$. These are the **regular points** of $X_\mathcal{D}$, and correspond in a natural way to the points of the D-ultrapower $X_\mathcal{D}$. In this way we may view $X_\mathcal{D}$ as a (necessarily dense) subset of $X_\mathcal{D}$. If $x \in X$, $x_\mathcal{D}$ denotes the regular point $\vec{x}_\mathcal{D}$, where $x_i = x$ for all $i \in I$. Clearly $p_{X,\mathcal{D}}(x_\mathcal{D}) = x$; more generally, $p_{X,\mathcal{D}}(\mu) = x$ if and only if $U^\mathcal{D}$ contains a closed-set ultraproduct $\vec{F}^\mathcal{D} \in \mu$ for every open neighborhood U of x.

In the case where the topology on X is induced by a linear order $\leq$, the ultrapower order $\leq^{\mathcal{D}}$, also linear, induces the subspace topology on $X_{\mathcal{D}}$. This fact is extremely useful in the study of ultra-arcs.

Regular closed sets $\vec{K}_{\mathcal{D}}$ where each K_i is a subcontinuum of X are called **regular subcontinua** of $X_{\mathcal{D}}$. For $\mu \in X_{\mathcal{D}}$, $\mathcal{R}(\mu)$ is the family of all regular subcontinua containing μ, and we define $\mu, \nu \in X_{\mathcal{D}}$ to be **$\mathcal{R}$-equivalent** if $\mathcal{R}(\mu) = \mathcal{R}(\nu)$. Clearly any $\mathcal{R}$-class containing a regular point is degenerate, so there are generally lots of $\mathcal{R}$-classes. The associated quotient map is denoted $r_{X,\mathcal{D}}: X_{\mathcal{D}} \to X_{\mathcal{D}}^{\mathcal{R}}$, and referred to as the **regularization map**.

The paper [12, Section 7] lays out a general theory of $\mathcal{R}$-classes in ultracoproduct continua. In particular, it gives conditions ensuring that $X_{\mathcal{D}}^{\mathcal{R}}$ is a continuum. What is of importance to us here is when $X = \mathbb{I}$, and we summarize below the salient facts (taken from the survey [18]) we will be using. We first define a continuum X to be **decomposable** if it is the union of two proper subcontinua; **indecomposable** otherwise. X is **unicoherent** if it is not the union of two subcontinua whose intersection is disconnected. Adding the modifier *hereditarily* to any descriptor confers the given property to all nondegenerate subcontinua.

Lemma 3.1

The following conditions hold for any ultra-arc $\mathbb{I}_{\mathcal{D}}$.

(i) $\mathbb{I}_{\mathcal{D}}$ *is irreducible about* {0D,1D}.

(ii) (Propositions 2.8 and 2.12, and Lemma 2.9, attributed to J. Mioduszewski) *The $\mathcal{R}$-classes–also known as* layers*–of $\mathbb{I}_{\mathcal{D}}$ are nowhere dense subcontinua, and at least one of them is nondegenerate.*

(iii) (Corollary 2.10, attributed to Mioduszewski) *With $\leq$ denoting the usual order on $\mathbb{I}$, the ultrapower order $\leq^{\mathcal{D}}$ induces a linear order on the set of layers, and the resulting* regularized $\mathcal{D}$-ultracopower $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$ *is a generalized arc.*

(iv) (iv) (Theorem 6.3, attributed to E. van Douwen, M. Smith, and J.-P. Zhu, independently) *All layers of $\mathbb{I}_{\mathcal{D}}$ are indecomposable subcontinua.*

(v) (Theorem 2.11, attributed to Mioduszewski) *Every subcontinuum of $\mathbb{I}_{\mathcal{D}}$ is either contained in a layer or is a union of layers.*

(vi) (Theorem 5.6, attributed to L. Gillman and M. Hendriksen) $\mathbb{H}^*$ *is hereditarily unicoherent (and therefore so is $\mathbb{I}_{\mathcal{D}}$).*

We end this section with two easy applications of Lemma 3.1. First recall that a continuous map $f: X \to Y$ between spaces is a **retraction** if f has a *right inverse*; i.e., a continuous $g: Y \to X$ with $f \circ g = \text{identity}_Y$.

Corollary 3.2

Ultra-arc regularization maps are monotone, but never retractions.

Proof

Ultra-arc regularization maps are monotone, by (ii). Suppose $g: \mathbb{I}_{\mathcal{D}}^{\mathcal{R}} \to \mathbb{I}_{\mathcal{D}}$ is a right-inverse for $r = r_{\mathbb{I},\mathcal{D}}$. By (i), plus the elementary fact that monotone maps preserve irreducibility, we know that $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$ is irreducible about $\{r(0_{\mathcal{D}}), r(1_{\mathcal{D}})\}$. Now $g(r(\mu)) = \mu$ whenever $\mu \in \mathbb{I}_{\mathcal{D}}$ is a regular point, hence the subcontinuum $K = g[\mathbb{I}_\mathcal{D}^\mathcal{R}]$ contains both $0_\mathcal{D}$ and $1_\mathcal{D}$. Consequently $K = \mathbb{I}_\mathcal{D}$, and r is a homeomorphism. But this implies all layers of $\mathbb{I}_\mathcal{D}$ are degenerate, contradicting (ii). □

We next show that ultra-arcs share an important property with generalized arcs.

Corollary 3.3

The ultra-arc $\mathbb{I}_\mathcal{D}$ is uniquely irreducible, with end points $0_\mathcal{D}$ and $1_\mathcal{D}$.

Proof

We already know that $\mathbb{I}_\mathcal{D}$ is irreducible about $\{0_\mathcal{D}, 1_\mathcal{D}\}$. Let $S \subseteq \mathbb{I}_\mathcal{D}$ be a closed set of irreducibility, and suppose, say, $0_\mathcal{D} \notin S$. Using the notation in the proof of Corollary 3.2, $r[S]$ cannot contain $r(0_\mathcal{D})$ because $r^{-1}[r(0_\mathcal{D})] = \{0_\mathcal{D}\}$. Because $\mathbb{I}_\mathcal{D}^\mathcal{R}$ is a generalized arc uniquely irreducible about $\{r(0_\mathcal{D}), r(1_\mathcal{D})\}$ (use (i) and (iii)), the closed set $r[S]$ is not a set of irreducibility for $\mathbb{I}_\mathcal{D}^\mathcal{R}$. Let K be a proper subcontinuum containing $r[S]$. Then $r^{-1}[K]$ is a proper subcontinuum of $\mathbb{I}_\mathcal{D}$ continuing S, a contradiction. □

4. Monotone images are decomposable

As mentioned above, monotone maps preserve irreducibility. They are also easily seen to preserve (hereditary) unicoherence. Thus every monotone image of an ultra-arc is irreducible and hereditarily unicoherent, by Lemma 3.1 (i, vi).

Remark 4.1

We do not know whether monotone images of ultra-arcs are *uniquely* irreducible. For while this is true for ultra-arcs by Corollary 3.3, the property is not preserved by monotone maps. Indeed, suppose X is decomposed as the union $A \cup Y \cup B$, where A and B are disjoint arcs, each sharing one of its end points with Y, and disjoint from Y otherwise. Let a (resp., y_A) be the end point of A not in (resp., shared with) Y; likewise identify b and y_B. Suppose further that Y is irreducible about $\{y_A, y_B\}$, but that Y is not uniquely irreducible. (E.g., Y is an indecomposable metric continuum.) Let $f: X \to Y$ now be the retraction that collapses A and B to $\{y_A\}$ and $\{y_B\}$, respectively. Then X is uniquely irreducible, unlike Y, and f is monotone.

Ultra-arcs are decomposable, but monotone maps need not preserve this property either. (Just consider the first coordinate projection from the product of an indecomposable continuum with a decomposable one.) In this section we address the issue of (hereditary) decomposability in monotone images of ultra-arcs.

A map whose domain is an ultracopower is **$\mathcal{R}$-preserving** if it sends two $\mathcal{R}$-equivalent points to the same point; i.e., if its point pre-images are unions of R-classes.

Lemma 4.2

Suppose $f: \mathbb{I}_\mathcal{D} \to X$ is an $\mathcal{R}$-preserving map. If either (a) X is nondegenerate and f is monotone, or (b) f is co-existential, then X is a generalized arc.

Proof

Since f is $\mathcal{R}$-preserving, there is a map $g: \mathbb{I}_\mathcal{D}^\mathcal{R} \to X$ such that $f = g \circ r_{\mathbb{I},\mathcal{D}}$. As a straightforward consequence of the definitions, we know that g is monotone (co-existential) if the same is true for f.

First assume X is nondegenerate and f is monotone. By Lemma 3.1 (i), we know X is irreducible about $\{f(0_{\mathcal{D}}), f(1_{\mathcal{D}})\}$. We show that every other point of X is a cut point.

So fix $x \in X \setminus \{f(0_{\mathcal{D}}), f(1_{\mathcal{D}})\}$. By Lemma 3.1 (iii), $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$ is a generalized arc. The subcontinuum $g^{-1}[x]$ does not contain either end point; hence there is a disconnection $\{U, V\}$ of $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}} \setminus g^{-1}[x]$ into disjoint nonempty open sets. Another appeal to monotonicity shows that $U = g^{-1}[g[U]]$ and $V = g^{-1}[g[V]]$. Also $g[U] \cap g[V] = \emptyset$; so $\{g[U], g[V]\}$ is a disconnection of $X \setminus \{x\}$, and we conclude that X is a generalized arc.

In the event f is a co-existential map, we know immediately that X is nondegenerate, and do not have to assume this. $\mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$, being a generalized arc, is automatically locally connected. This property in compacta is preserved by continuity, so X is locally connected too. But, by Lemma 2.1, f is now monotone, and we argue as above. □

Proposition 4.3

Let X be a metric continuum. Then X is an arc if and only if X is the image (resp., nondegenerate and the image) of some ultra-arc under an $\mathcal{R}$-preserving map that is co-existential (resp., monotone).

Proof

To conclude X is an arc, just apply Lemma 4.2. For the converse, note that the codiagonal map $p_{\mathbb{I},\mathcal{D}}: \mathbb{I}_{\mathcal{D}} \to \mathbb{I}$ is co-elementary; and also monotone, by Lemma 2.1. Point pre-images always contain many regular points; hence, by Lemma 3.1 (v), they are unions of layers. Thus $p_{\mathbb{I},\mathcal{D}}$ is $\mathcal{R}$-preserving. □

Lemma 4.4

Suppose $f: \mathbb{I}_{\mathcal{D}} \to X$ is a monotone surjection which is not $\mathcal{R}$-preserving. Then X contains a nondegenerate indecomposable subcontinuum.

Proof

Since f is not $\mathcal{R}$-preserving, there is a layer $R \subseteq \mathbb{I}_{\mathcal{D}}$ which is not contained in any point pre-image under f. Since f is monotone, we know from Lemma 3.1 (v) that for any $x \in X$, $f^{-1}[x]$ is either disjoint from R or properly contained in R. Let $K = \{x \in X: f^{-1}[x] \subseteq R\}$. Then $R = f^{-1}[K]$ and K has more than one point. By Lemma 3.1 (ii), then, K is a nondegenerate subcontinuum of X. Now $f|R: R \to K$ is a monotone map because R is a union of point pre-images. Since R is an indecomposable continuum (Lemma 3.1 (iv)), we conclude that K is a nondegenerate indecomposable subcontinuum of X. □

Theorem 4.5

Let X be a nonlinear continuum which is a monotone image of some ultra-arc. Then X contains a nondegenerate indecomposable subcontinuum.

Proof

Suppose $f: \mathbb{I}_{\mathcal{D}} \to X$ is a monotone surjection, where X is nonlinear. We use Lemma 4.2 to infer that f is not $\mathcal{R}$-preserving. Now apply Lemma 4.4 to infer that X contains a nondegenerate indecomposable subcontinuum. □

Remark 4.6

In Theorem 4.5, *monotone* cannot be replaced with *co-existential*: The $\sin\left(\frac{1}{x}\right)$-continuum (see [26]) is a nonlinear hereditarily decomposable continuum, and hence not a monotone image of any ultra-arc. It is, however, chainable; and–by Theorem 6.3 below–thus a co-existential image of each ultra-arc.

Of course an ultra-arc is nonlinear and a monotone image of itself. But this raises the question of what happens in the metric realm.

Question 4.7

If X is a nondegenerate monotone metric image of an ultra-arc, is X necessarily an arc?

Finally we consider decomposability in monotone images.

Theorem 4.8

A nondegenerate monotone image of an ultra-arc is decomposable.

Proof

Suppose $f: \mathbb{I}_{\mathcal{D}} \to X$ is a monotone map onto a nondegenerate continuum X. For $n = 0,1$, let $x_n = f(n_{\mathcal{D}})$ and $K_n = f^{-1}[x_n]$. If L and R are any layers in $\mathbb{I}_{\mathcal{D}}$, extend interval notation by letting $[L, R]$ denote the union of all layers lying between L and R in the pre-order induced by the ultrapower order $\leq^{\mathcal{D}}$ (see Lemma 3.1, also [18, Theorem 2.11]). Then $[L, L] = L$; and if $L \neq R$, we have $[L, R]$ equal to the interval $[\mu, \nu]$, where $\mu \in L$ and $\nu \in R$. Now every subcontinuum not properly contained in a layer is of the form $[L, R]$, where L and R are layers. Since regular points determine degenerate layers, we have $K_0 = [0_{\mathcal{D}}, L_0]$ and $K_1 = [L_1, 1_{\mathcal{D}}]$, for some layers L_0 and L_1. Since L0≠L1 and X is irreducible about $\{x_0, x_1\} = \{f(L_0), f(L_1)\}$, we know $[L_0, L_1]$ is a decomposable subcontinuum of $\mathbb{I}_{\mathcal{D}}$ which maps via f onto X. Also $[L_0, L_1]$ is irreducible between any point of L_0 and any point of L_1; hence any proper subcontinuum $K \subseteq [L_0, L_1]$ is disjoint from at least one of $\{K_0, K_1\}$, and therefore maps to a proper subcontinuum of X. Since $[L_0, L_1]$ is decomposable, so too is X. □

Question 4.9

If X is nondegenerate and $f: \mathbb{I}_{\mathcal{D}} \to X$ is a monotone surjection, is f co-existential? Alternatively: is a nondegenerate monotone image of an ultra-arc necessarily a co-existential image?

5. Generalized arcs which are co-elementary monotone images

We saw earlier that the codiagonal maps $p_{\mathbb{I},\mathcal{D}}$ witness the fact that an arc is a co-elementary image of any ultra-arc. Because arcs are locally connected, these maps are monotone as well. In this section we show that every generalized arc of weight $\aleph_1$ is also a co-elementary image of any ultra-arc.

The following mapping existence theorem serves as our main lemma for proving that certain nonmetric continua are special images of ultra-arcs. Its proof uses classical results from saturated model theory, so we point out only the highlights of the argument. We note first that while in model theory the principal cardinal invariant of a structure is the cardinality of its underlying set, the corresponding invariant for a compactum is its weight. (So what corresponds to countable cardinality in a structure is metrizability in a compactum.)

Lemma 5.1

Let X be a compactum of weight $\leq \aleph_1$ and $\mathcal{D} \in \omega^*$. Then there is a metric compactum Y and co-elementary maps $f: X \to Y$, $g: Y_{\mathcal{D}} \to X$ such that $f \circ g = p_{Y,\mathcal{D}}$.

Proof

We assume all spaces to be infinite; otherwise there is nothing to prove.

First we note that a **lattice base** for a compactum Y is a closed-set base that is itself closed under finite unions and intersections (see [7]). What is most important for us about lattice bases of compacta is due to H. Wallman [29]: an abstract structure B over the lexicon of bounded lattices is isomorphic to a lattice base for a compactum if and only if B is a normal disjunctive distributive lattice. These lattice conditions may be expressed as first-order sentences; thus the class of lattice bases for compacta is an elementary class (in the sense of [15]). So if $\mathcal{D}$ is an ultrafilter on index set I and $\mathcal{B}$ is a lattice base for compactum Y, then the $\mathcal{D}$-ultrapower $\mathcal{B}^{\mathcal{D}}$ is (naturally isomorphic to) a lattice base for the $\mathcal{D}$-ultracopower $Y_{\mathcal{D}}$.

Now suppose X is a compactum of weight $\leq \aleph_1$, and fix a lattice base $\mathcal{A}$ for X, where $|\mathcal{A}| \leq \aleph_1$. By the Löwenheim-Skolem Theorem [15, Theorem 3.1.6], there is a countable elementary sublattice $B \subseteq \mathcal{A}$. Enumerate the elements of B, and regard $\langle B, b \rangle_{b \in B}$ as a relational structure in the lexicon of bounded lattices, enhanced with a constant symbol for each $b \in B$. Then $\langle B, b \rangle_{b \in B}$ is an elementary substructure of $\langle \mathcal{A}, b \rangle_{b \in B}$.

By the elementarity feature inherent in Wallman's theorem, there is a compactum Y having a countable lattice base $\mathcal{B}$ isomorphic to B. Fix such an isomorphism, and for each $b \in B$, let $B_b \in \mathcal{B}$ correspond to b. Then there is an elementary embedding φ: $\langle \mathcal{B}, B_b \rangle_{b \in B} \to \langle \mathcal{A}, b \rangle_{b \in B}$.

Let $\mathcal{D} \in \omega^*$. Then, because $\mathcal{D}$ is countably incomplete and $\aleph_1$-good (see [15, Exercises 6.1.2 and 6.1.5]), the ultrapower $\langle \mathcal{B}, B_b \rangle_{b \in B}^{\mathcal{D}}$ is $\aleph_1$-saturated [15, Theorem 6.1.8]. By [15, Theorem 5.1.2], this makes $\langle \mathcal{B}, B_b \rangle_{b \in B}^{\mathcal{D}}$ $\aleph_2$-universal; i.e., any structure of cardinality $\leq \aleph_1$ which is elementarily equivalent to it is elementarily embeddable in it. Since $\langle \mathcal{A}, b \rangle_{b \in B}$ is just such a structure, we have an elementary embedding $\gamma: \mathcal{A} \to \mathcal{B}^{\mathcal{D}}$ such that $\gamma \circ \varphi$ is the natural diagonal embedding from $\mathcal{B}$ to $\mathcal{B}^{\mathcal{D}}$. By applying the maximal spectrum functor, we obtain co-elementary maps $f: X \to Y$ and $g: Y_{\mathcal{D}} \to X$ (induced by φ and γ, respectively) so that $f \circ g = p_{Y,\mathcal{D}}$. □

Theorem 5.2

Every generalized arc of weight $\leq \aleph_1$ is a co-elementary monotone image of any ultra-arc.

Proof

Let X be a generalized arc of weight $\leq \aleph_1$, with $\mathcal{D}$ any free ultrafilter on ω. Let $f: X \to Y$ and $g: Y_{\mathcal{D}} \to X$ be as in Lemma 5.1. (We do not need the commutativity condition here.) Since f is co-elementary (just co-existential will do), we know that Y is an arc [4, Proposition 2.3]. Thus $Y_{\mathcal{D}}$ is an ultra-arc that co-elementarily maps, via g, onto X. This map is monotone, by Lemma 2.1. □

Remark 5.3

From Lemma 3.1 (ii, iii), the regularization map $r_{\mathbb{I},\mathcal{D}}: \mathbb{I}_{\mathcal{D}} \to \mathbb{I}_{\mathcal{D}}^{\mathcal{R}}$ is a monotone map from an ultra-arc onto a generalized arc of weight c. However, we do not know whether this map is co-existential. (See [4, Proposition 2.7], where a fair amount of effort is expended in showing that a monotone map from one arc onto another is always co-elementary.)

6. Chainable continua which are co-existential images

A continuum X is **chainable** if each open cover of X refines to a finite open cover $\{U_1, \dots, U_n\}$, ordered in such a way that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. From this definition it is trivial to see that nondegenerate chainable continua are **one-dimensional** (in the covering sense); i.e., no point lies in more than two members of the refined cover.

Remark 6.1

Covering dimension is preserved under the taking of ultracopowers [3, Theorem 2.2.2], and co-existential maps (unlike monotone maps [34, Theorem 1]) cannot raise covering dimension [6, Theorem 2.6]. This tells us that ultra-arcs are one-dimensional, and hence so are all their co-existential–but perhaps not necessarily monotone–images. The picture is quite different with chainability: nondegenerate ultracopowers via countably incomplete ultrafilters are never chainable [1, Lemma 5.3], so the fact that chainability is preserved by co-existential maps [10, Section 5] is not of any use. The question naturally arises as to whether all metric co-existential images of ultra-arcs are chainable, and in the next section we give a negative answer.

It is easy to show that generalized arcs are both chainable and locally connected, but the converse is only known to be true for metric continua (see [26, Chapter XII]). Since continuous maps between compacta preserve local connectedness and co-elementary (even co-existential) maps preserve chainability, the proof of Theorem 5.2 may easily be adapted to prove the following.

Corollary 6.2

If X is a nondegenerate locally connected chainable continuum of weight $\leq \aleph_1$, then X is a co-elementary monotone image of every ultra-arc.

It is well known [26, Theorems 12.11, 12.19] that a nondegenerate metric continuum is chainable if and only if it is an inverse limit of an ω-sequence of arcs and continuous surjections for bonding maps.

The following is proved in far more generality as Theorem 1.2 (ii) (and Corollary 2.1) in [8]. This special case holds the most interest, however, and its proof is much clearer. Thus we include it here.

Theorem 6.3

Every nondegenerate chainable metric continuum is a co-existential image of any ultra-arc.

Proof

Fix ultra-arc $\mathbb{I}_{\mathcal{D}}$, and let X be a nondegenerate chainable metric continuum. Then we may view X as the inverse limit of an ω-sequence $\vec{f}$ of continuous surjections on $\mathbb{I}$:

$$\mathbb{I} \xleftarrow{f_0} \mathbb{I} \xleftarrow{f_1} \mathbb{I} \xleftarrow{f_2} \dots$$

Let $\vec{\pi}$ be the associated sequence of projection maps from X to $\mathbb{I}$, so that $\pi_n = f_n \circ \pi_{n+1}$ always holds. The $\mathcal{D}$**-ultracoproduct map** $\vec{\pi}_{\mathcal{D}}: X_{\mathcal{D}} \to \mathbb{I}_{\mathcal{D}}$ of this sequence is defined by the rule:

$$\prod_{\mathcal{D}} F_n \in \vec{\pi}_{\mathcal{D}}(\mu) \text{ iff } \prod_{\mathcal{D}} \vec{\pi}_n^{-1}[F_n] \in \mu.$$

(See [3], where a more general definition is given to cover ultracoproducts of noncompact spaces.) $\vec{\pi}_{\mathcal{D}}$ is a continuous surjection because the same is true for each π_n.

For each $n \in \omega$, define $\gamma_n: \mathbb{I} \times (\omega \setminus n) \to \mathbb{I}$ via the rule $\gamma_n(t, m) = f_m(t)$ for $t \in \mathbb{I}$ and $m \geq n$. Since each terminal segment $\omega \setminus n$ is in $\mathcal{D}$, we have $\mathbb{I}_{\mathcal{D}} \subseteq \beta(\mathbb{I} \times (\omega \setminus n))$. Define $g_n: \mathbb{I}_{\mathcal{D}} \to \mathbb{I}$ to be the restriction of γ_n^{β} to $\mathbb{I}_{\mathcal{D}}$.

It is straightforward to check that the commutativities $g_n \circ \vec{\pi}_{\mathcal{D}} = \pi_n \circ p_{X,\mathcal{D}}$ and $g_n = f_n \circ g_{n+1}$ always hold. Hence, by basic properties of inverse limits, there is a unique $g: \mathbb{I}_{\mathcal{D}} \to X$ so that $g \circ \vec{\pi}_{\mathcal{D}} = p_{X,\mathcal{D}}$ also holds. This shows directly that g is a co-existential map. □

An obvious question at this point is whether the metric hypothesis in Theorem 6.3 can be weakened to being of weight $\leq \aleph_1$, and the answer is a qualified *yes*.

Given $\mathcal{D}, \mathcal{E} \in \omega^*$, denote by $\mathcal{D} \cdot \mathcal{E}$ the **Fubini product** of the two ultrafilters, itself an ultrafilter on $\omega \times \omega$, as follows: For $R \subseteq \omega \times \omega$, and $i \in \omega$, denote by $R(i) := \{j \in \omega: \langle i, j \rangle \in R\}$. Then R is a member of $\mathcal{D} \cdot \mathcal{E}$ if and only if $\{i \in \omega: R(i) \in \mathcal{E}\} \in \mathcal{D}$. Free ultrafilter $\mathcal{F}$ is a **Fubini ultrafilter** if there is a bijection between ω and $\omega \times \omega$ which induces an equivalence between $\mathcal{F}$ and $\mathcal{D} \cdot \mathcal{E}$ for some $\mathcal{D}, \mathcal{E} \in \omega^*$. A **Fubini ultra-arc**, then, is just an ultra-arc whose indexing ultrafilter is Fubini.

Theorem 6.4

Every chainable continuum of weight $\aleph_1$ is a co-existential image of any Fubini ultra-arc.

Proof

Assume X is a chainable continuum of weight $\aleph_1$, with $\mathcal{F} = \mathcal{D} \cdot \mathcal{E}$ an arbitrary Fubini ultrafilter. Using Lemma 5.1, we have a metric continuum Y and co-elementary maps $f: X \to Y$, $g: Y_{\mathcal{D}} \to X$. (We do not need the commutativity condition here either.) Y is nondegenerate, and–as stated in Remark 6.1–chainable. Thus, by Theorem 6.3, there is a co-existential map $k: \mathbb{I}_{\mathcal{E}} \to Y$. Let $k_{\mathcal{D}}: (\mathbb{I}_{\mathcal{E}})_{\mathcal{D}} \to Y_{\mathcal{D}}$ be the $\mathcal{D}$-ultracopower of k. Then [5, Corollary 2.4] kD is co-existential; moreover [3, Theorem 2.1.1], there is a homeomorphism $h: \mathbb{I}_{\mathcal{F}} \to (\mathbb{I}_{\mathcal{E}})_{\mathcal{D}}$. Compositions of co-existential maps are co-existential [5, Proposition 2.5], so $g \circ k_{\mathcal{D}} \circ h: \mathbb{I}_{\mathcal{F}} \to X$ is our desired co-existential map. □

Question 6.5

Is the *Fubini* assumption necessary in Theorem 6.4?

There is a *no* answer to this if we assume the Continuum Hypothesis (CH: $\mathfrak{c} := 2^{\aleph_0} = \aleph_1$). The following is a corollary of Theorem 6.4, modulo some standard model-theoretic arguments.

Corollary 6.6

(CH) *Every nondegenerate chainable continuum of weight* $\leq \aleph_1$ *is a co-existential image of any ultra-arc.*

Proof

Let $\mathcal{A}$ be a countable lattice base for $\mathbb{I}$. Given $\mathcal{D}, \mathcal{E} \in \omega^*$, the ultrapowers $\mathcal{A}^{\mathcal{D}}$ and $\mathcal{A}^{\mathcal{E}}$ are both $\aleph_1$-saturated [15, Theorem 6.1.8] and of cardinality c. Since $\mathfrak{c} = \aleph_1$ and the ultrapowers are elementarily equivalent, they must be isomorphic [15, Theorem 5.1.13]. This immediately gives us that any two ultra-arcs are homeomorphic. So if X is a chainable continuum of weight $\aleph_1$, then Theorem 6.4 tells us X is a co-existential image of *some* ultra-arc. Hence it is a co-existential image of *every* ultra-arc. □

Remarks 6.7

(i) One way to get an absolute (in ZFC) *no* answer to Question 6.5 would be to show that every ultra-arc is homeomorphic to a Fubini ultra-arc. While we do not know whether this can be done, it cannot be done by trying to show every ultrafilter is Fubini: a Fubini ultrafilter is not a weak P-point in ω^* (i.e., one not in the closure of any countable subset of its complement), and there are $2^{\mathfrak{c}}$ weak P-points in ω^* [22, Theorem 0.1].

(ii) In Theorem 5.2, Theorem 6.4 only the existence of co-elementary maps f and g is used from Lemma 5.1, and the commutativity $f \circ g = p_{Y,\mathcal{D}}$ is never required. We feel there is much potential left in Lemma 5.1.

7. Nonchainable continua which are co-existential images

As mentioned in Remark 6.1, chainability is preserved by co-existential maps, but ultra-arcs are not chainable. So it makes sense–in the interests of balancing the results of the last section–to ask whether there are nonchainable metric co-existential images of ultra-arcs.

A continuum X is **co-existentially closed** if whenever $f: Y \to X$ is a continuous map from a continuum onto X, then f is co-existential.

Remark 7.1

The notion *co-existentially closed continuum* is dual to that of *existentially closed model of a universal theory* in mathematical logic, and is analogous to the well-studied properties Class (C) ("confluently closed continuum") and Class (W) ("weakly-confluently closed continuum"). See, e.g., [26] for details. Note that there is no such thing as a nondegenerate *monotonically closed continuum* because every nondegenerate continuum is easily seen to be a continuous image of another continuum under a nonmonotone map. Also there are no *co-elementarily closed continua* at all. This is true because: (i) co-elementary maps preserve covering dimension [7, Theorem 5.5 (1)]; and (ii) every continuum is a continuous image of a continuum of different dimension. (From [7, Theorems 5.20, 5.22] we know that every continuum is a continuous image of a one-dimensional one; any one-dimensional continuum is a continuous image of an infinite-dimensional one.)

The class of co-existentially closed continua, however, is quite substantial.

Lemma 7.2

(i) *[6, Theorem 6.1] Every continuum is a continuous image of a co-existentially closed continuum of the same weight.*

(ii) *[8, Corollary 4.13] Every co-existentially closed continuum is hereditarily indecomposable and one-dimensional.*

Since there are co-existentially closed continua of any given weight, Theorem 1.1 (ii) may be meaningfully applied.

Corollary 7.3

Every co-existentially closed continuum of weight $\leq \aleph_1$ is a co-existential image of any ultra-arc.

Remark 7.4

It is worth noting that since a co-existentially closed continuum is nondegenerate and indecomposable (Lemma 7.2 (ii)), it cannot be a monotone image of any ultra-arc (Theorem 4.8). Nor can it be a co-elementary image, because ultra-arcs are decomposable and co-elementary maps both preserve and reflect this property (see, e.g., [7, Theorem 5.5]).

Of the nondegenerate hereditarily indecomposable metric continua, the *pseudo-arc* $\mathbb{P}$ is unique for being chainable (see the survey [23]). Thus $\mathbb{P}$ is a co-existential image of every ultra-arc, by Theorem 6.3. But it is a co-existential image for another reason, because we may apply Corollary 7.3 to the following.

Theorem 7.5

[17, Main Theorem] P *is a co-existentially closed continuum.*

Indeed, when we add Lemma 7.2 (ii), we see that the pseudo-arc is the only co-existentially closed metric continuum which is chainable. There are plenty of nonchainable ones too, it turns out.

Theorem 7.6

There is a family of continuum-many topologically distinct co-existentially closed metric continua which are not chainable.

Proof

In [30], Z. Waraszkiewicz constructs a family $\mathcal{S}$ of $\mathfrak{c}$ pairwise nonhomeomorphic subcontinua of the Euclidean plane, such that no metric continuum continuously maps onto more than countably many members of $\mathcal{S}$. Using Lemma 7.2 (i), pick, for each $S \in \mathcal{S}$, a co-existentially closed metric continuum X_S which continuously maps onto S. Then each point pre-image under the assignment $S \mapsto X_S$ is countable; hence there must be c point pre-images. Thus there is a family $\mathcal{C}$ of $\mathfrak{c}$ topologically distinct metric co-existentially closed continua. Using Lemma 7.2 (ii), plus the topological characterization of $\mathbb{P}$, at most one member of $\mathcal{C}$ can be chainable. □

So applying Corollary 7.3 gives us the following.

Corollary 7.7

There is a family of continuum-many topologically distinct metric continua which are co-existential images of any ultra-arc, but which are not chainable.

Remarks 7.8

(i) A continuum X is **span zero** if whenever Y is a continuum and $f, g: Y \to X$ are continuous maps such that $f[Y] \subseteq g[Y]$, it follows that $f(y) = g(y)$ for some $y \in Y$. Chainable metric continua are well known to be span zero, but the converse is not true, even for continua in the plane [19]. However, any hereditarily indecomposable metric continuum is chainable if it is span zero [20, Theorem 1]; thus *chainable* may be replaced in Corollary 7.7 with *span zero*. We do not know whether a metric continuum of nonzero span can be a co-existential image of an ultra-arc without being hereditarily indecomposable.

(ii) As an immediate consequence of Theorem 1.1 (ii) and Lemma 7.2 (i), there is a co-existentially closed continuum X, of weight $\mathfrak{c}$, which continuously maps onto each continuum of weight $\leq \aleph_1$.

8. Nonlinear images and local connectedness

In this section we consider the following question: if a nonlinear continuum is a special image of an ultra-arc, how close to being locally connected can it be? Our conclusion is: *not very*.

To make this a bit clearer, we identify two properties; the first stronger than the second, and both weaker than local connectedness for continua. The stronger is *aposyndesis*, initially studied by F. B. Jones (see [21]): A topological space X is **aposyndetic** if for each two points of X, one of them is contained in the interior of a closed connected subset that excludes the other. Aposyndesis clearly follows from local connectedness in regular T_1 spaces and is a kind of separation property. The suspension over the compact ordinal space $\omega + 1$ is well known to be aposyndetic, but not locally connected.

The second property, somewhat less well known than aposyndesis, is what we label *antisymmetry* in [11]: a space X is **antisymmetric** if for any triple $\langle a, b, c \rangle$ in X with $b \neq c$, there is a closed connected subset containing a and exactly one of b, c. With point a serving as "base point" and defining $x \leq_a y$ to mean that every closed connected subset containing a and y contains x as well, it is easily seen that X is antisymmetric if and only if each $\leq_a$ is antisymmetric as a pre-order.

Antisymmetry was also studied, in the context of metric continua, by B. E. Wilder [32], who called it "Property C."

Lemma 8.1

Every connected aposyndetic space is antisymmetric.

Proof

Let $\langle a, b, c \rangle$ be a triple in the connected space X, with $b \neq c$. We first claim that there is a connected set $A \subseteq X$ which contains a and exactly one of b, c. Indeed, suppose C is the component of $X \setminus \{b\}$ that contains c. If $a \in C$, we put $A = C$ and obtain a connected set containing a and c, but not b. In the case $a \notin C$, noting that both X and $\{b\}$ are connected, we use [27, Theorem 3.3] to infer that $A = X \setminus C$ is a connected set containing a and b, but not c.

Now suppose $A \subseteq X$ is a connected set containing a and b, but not c. For each $x \in A$, use aposyndesis to find open set U_x and connected closed set K_x with $x \in U_x \subseteq K_x \subseteq X \setminus \{c\}$. $\{U_x: x \in A\}$ is an open cover of the connected set A; hence there is a finite subfamily $\{U_{x_1}, \dots, U_{x_n}\}$, where $a \in U_{x_1}$, $b \in U_{x_n}$, and $U_{x_i} \cap U_{x_{i+1}} \neq \emptyset$ for $1 \leq i \leq n-1$. Then $K_{x_1} \cup \dots \cup K_{x_n}$ is a connected closed set containing a and b, but not c. □

Remarks 8.2

(i) The cone over $\omega + 1$, also known as the *harmonic fan*, is an antisymmetric continuum which is not aposyndetic.

(ii) The version of Lemma 8.1 in which X is a metric continuum was first proved by Wilder [32, Theorem 1], who also observed that nondegenerate antisymmetric metric continua are decomposable. We do not know whether this is still true in the nonmetric case, as it relies on the fact that indecomposable metric continua are irreducible.

(iii) The connectedness assumption in Lemma 8.1 is essential: indeed, only the two-point discrete space is both disconnected and antisymmetric.

Given points a, b in a continuum X, it is convenient to define the **interval** $[a, b]$ to constitute the intersection of all subcontinua that contain both a and b. In interval terms, then, antisymmetry is the statement that if $c \in [a, b]$ and $b \in [a, c]$, then $b = c$.

In [25] an **arboroid** is a hereditarily unicoherent continuum X which has the feature that each doubleton subset is the set of end points of a generalized arc in X. (So a dendroid–see [26]–is a metric arboroid.)

Lemma 8.3

[11, Corollary 4.8] A continuum is an arboroid if and only if it is antisymmetric, and each of its nondegenerate intervals has at least three points.

Theorem 8.4

Suppose X is an antisymmetric monotone image of some ultra-arc. Then X is a generalized arc.

Proof

Let $f: \mathbb{I}_{\mathcal{D}} \to X$ be a monotone map from an ultra-arc onto a nondegenerate antisymmetric continuum X. Since $\mathbb{I}_{\mathcal{D}}$ is hereditarily unicoherent (Lemma 3.1 (vi)), and monotone maps are easily seen to preserve this property, we know that X is also hereditarily unicoherent. It is easy to show that in hereditarily unicoherent continua, intervals are subcontinua. Hence, by Lemma 8.3, X is an arboroid. By Lemma 3.1 (i) and the fact that f is monotone, X is irreducible about the two points $f(0_{\mathcal{D}}), f(1_{\mathcal{D}})$. But then $X = [f(0_{\mathcal{D}}), f(1_{\mathcal{D}})]$ is a generalized arc. □

In the interests of obtaining a companion to Theorem 8.4 for co-existential maps, we first need to address the preservation of hereditary unicoherence.

Lemma 8.5

[9, Theorem 5.3] Hereditary unicoherence is preserved by co-existential maps between continua.

So combining Lemma 8.5 with the proof of Theorem 8.4, we obtain the following.

Corollary 8.6

Suppose X is an antisymmetric continuum that is a co-existential image of some ultra-arc. Then X is an arboroid.

Remark 8.7

What stands in the way of concluding in Corollary 8.6 that X is a generalized arc is that co-existential maps–even co-elementary ones–fail in general to preserve irreducibility. (See [12, Theorem 4.7] and surrounding discussion.) We address in Corollary 8.15 below the question of whether co-existential images are irreducible in the special case where the domain is an ultra-arc.

We next show that when *monotone* is replaced with *co-existential* in Theorem 8.4, we can conclude that X is a generalized arc, as long as we assume it to be aposyndetic.

First define X to be **connected *im kleinen*** at point $x \in X$ if every open neighborhood of x contains a subcontinuum which in turn contains x in its interior. A continuum is locally connected if and only if it is connected *im kleinen* at each of its points [33].

The following was first proved in [14] for metric continua, but the argument may be extended to all continua by means of Zorn's Lemma.

Lemma 8.8

Every hereditarily unicoherent aposyndetic continuum is locally connected.

Proof

Given an open set $U \subseteq X$ and $x \in U$, we say x is **aposyndetic in** U to mean that x is in the interior $\mathrm{int}(M) := \mathrm{int}_X(M)$ of a subcontinuum $M \subseteq U$. (So X is connected *im kleinen* at x if and only if x is aposyndetic in each of its open neighborhoods.)

Fix $x \in X$ and let $\mathcal{U}$ be the collection of open neighborhoods in which x is not aposyndetic. Assuming $\mathcal{U}$ to be nonempty and partially ordered by set inclusion, with $\mathcal{V}$ a linearly ordered subcollection, let $V = \bigcup \mathcal{V}$. If x were aposyndetic in $\mathcal{V}$, then we would have a subcontinuum $M \subseteq V$ with $x \in \mathrm{int}(M)$. But $\mathcal{V}$ is a linearly ordered open cover of M; hence $M \subseteq U$ for some $U \in \mathcal{V}$. This makes x aposyndetic in a member of $\mathcal{U}$, a contradiction. Hence, by Zorn's Lemma, $\mathcal{U}$ has a maximal element.

Now suppose X is a hereditarily unicoherent aposyndetic continuum. If X is not locally connected, then we may fix some $x \in X$ at which X is not connected *im kleinen*. By the paragraph above there is an open neighborhood U of x which is maximal subject to the condition that x fails to be aposyndetic in U. Since x is trivially aposyndetic in X, we know U is proper. Fix $y \in X \setminus U$. Since X is aposyndetic, there is a subcontinuum $M \subseteq X \setminus \{y\}$ with $x \in int(M)$.

Let $A = M \setminus U$. Then $X \setminus A$ is an open set containing U. Furthermore, $X \setminus A$ properly contains U because $y \in (X \setminus A) \setminus U$. Hence x is aposyndetic in $X \setminus A$, and we may find a subcontinuum $N \subseteq (X \setminus A)$ such that $x \in \mathrm{int}(N)$. Now $x \in \mathrm{int}(M) \cap \mathrm{int}(N) \subseteq \mathrm{int}(M \cap N) \subseteq M \cap N \subseteq M \cap (X \setminus A) = M \setminus (M \setminus U) \subseteq U$. Since x is not aposyndetic in U, we infer that $M \cap N$ is not connected. But this contradicts hereditary unicoherence. □

We can now state our first companion to Theorem 8.4.

Theorem 8.9

Suppose X is an aposyndetic co-existential image of some ultra-arc. Then X is a generalized arc.

Proof

Let $f: \mathbb{I}_{\mathcal{D}} \to X$ be a co-existential map from an ultra-arc onto an aposyndetic continuum X.
Using Lemma 8.1, Lemma 8.5, we argue–as in the proof of Theorem 8.4–that X is an arboroid. But now we invoke Lemma 8.8 to conclude that X is actually locally connected. By Lemma 2.1, f is monotone, and X is now irreducible. This makes X a generalized arc. □

Our second companion to Theorem 8.4 allows the assumption of antisymmetry, but only in the metric context. The obstruction is that we do not know whether co-existential images of ultra-arcs are irreducible in general; however, in the metric case, we have an affirmative answer, thanks to a deep theorem of R. Sorgenfrey.

Recall that a continuum X is a **triod** if there is a subcontinuum $K \subseteq X$ such that $X \setminus K$ partitions into three open sets. X is a **weak triod** if there is a cover of X by three subcontinua, no one of which is contained in the union of the other two.

Remark 8.10

If $\langle K, U_1, U_2, U_3 \rangle$ witnesses that X is a triod, then each U_i is clopen in $X \setminus K$. Hence, by [27, Theorem 3.4], $K \cup U_i$ is a subcontinuum of X, and we have $\langle K \cup U_1, K \cup U_2, K \cup U_3 \rangle$ witnessing that X is a weak triod.

Lemma 8.11

([26, Theorem 11.34], attributed to R. Sorgenfrey) *Every nondegenerate unicoherent metric continuum which is not a triod is irreducible.*

Remark 8.12

The metric assumption in Lemma 8.11 cannot be discarded: D. Bellamy [13] has constructed a continuum of weight $\aleph_1$ which is indecomposable, with but a single composant. It is hence unicoherent and not a triod, but still not irreducible.

The most important feature of co-existential maps for our present purpose is that they are *weakly confluent* (i.e., subcontinua of the range are images of subcontinua of the domain) in a particularly uniform way. The following is an immediate corollary of [6, Theorem 2.4].

Lemma 8.13

Let $f: Y \to X$ be a co-existential map between compacta. Then there is a $\cup$-preserving homomorphism f^* from the subcompacta of X to the subcompacta of Y such that for any subcompactum $F \subseteq X$: (i) $f[f^*(F)] = F$; (ii) $f^{-1}[U] \subseteq f^*(F)$ whenever $U \subseteq F$ is open in X; and (iii) $f^*(K)$ is a subcontinuum of Y for any subcontinuum $K \subseteq X$.

From this fact, we may now prove the following.

Lemma 8.14

Co-existential maps reflect being a weak triod.

Proof

Let $f: Y \to X$ be co-existential, and assume X is a weak triod. Then we have a triple $\langle K_1, K_2, K_3 \rangle$ of subcontinua of X, where $K_1 \cup K_2 \cup K_3 = X$ and $K_i \setminus (K_j \cup K_l) \neq \emptyset$ whenever $i \notin \{j, l\}$.

Let f^* now be the set map in Lemma 8.13, and set $M_i = f^*(K_i)$, $i = 1,2,3$. By clause (iii) each M_i is a subcontinuum of Y. Since f^* preserves finite unions, $M_1 \cup M_2 \cup M_3 = f^*(X) = f^{-1}[X] = Y$, by clause (ii).

Suppose, say, that $M_1 \subseteq M_2 \cup M_3$. Then, by clause (i), $K_1 = f[M_1] \subseteq f[M_2] \cup f[M_3] = K_2 \cup K_3$, a contradiction. Hence the triple $\langle M_1, M_2, M_3 \rangle$ witnesses that Y is a weak triod. □

Corollary 8.15

Any co-existential metric image of a unicoherent irreducible continuum is irreducible. In particular, any co-existential metric image of an ultra-arc is irreducible.

Proof

Let $f: Y \to X$ be co-existential, where Y is unicoherent and irreducible. Y is then clearly not a weak triod; so by Lemma 8.14, neither is X. Co-existential maps preserve unicoherence. Thus if we further assume X is metric, we may use Lemma 8.11 to conclude that X is irreducible. □

Remark 8.16

A co-existential metric image of an ultra-arc need not be *uniquely* irreducible (see Remark 4.1): By Theorem 6.3, we obtain a host of indecomposable examples, as well as the hereditarily decomposable $\sin\left(\frac{1}{x}\right)$-continuum.

Combining Corollary 8.15 with Lemma 8.5 and the argument proving Theorem 8.4 immediately gives us our second variant of Theorem 8.4.

Theorem 8.17

Suppose X is a metric continuum which is an antisymmetric co-existential image of some ultra-arc. Then X is an arc.

Question 8.18

If a nondegenerate arboroid is not a triod, must it be a generalized arc?

The answer to this question is positive in the metric case; a positive answer in general would allow us to conclude *generalized arc* outright in Corollary 8.6, obviating the need for Sorgenfrey's Theorem (Lemma 8.11), as well as any of the results above mentioning aposyndesis.

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