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Dissipative Resilient Observer

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Abstract:

Cybersecurity is a major concern for designers of control systems that can be directed against any of their components. Observers are an integral part of control systems that require state feedback. This paper considers an observer subject to errors in implementation or subject to cyberattacks. The errors and cyberattacks result in perturbations in the gain and in a finite-energy but unknown disturbance input. We obtain conditions for Q-S-R dissipativity and stability of the observer in the presence of the gain errors and disturbances in the form of linear matrix inequalities (LMIs). Three examples are presented to show how the LMIs can yield resilient observer designs.

SECTION I. Introduction

Observers form an important component of many control systems where the state estimate is needed to provide feedback control. **Because** of errors in implementation and the threat of cyberattacks, observers can cease to function properly, which can result in unacceptable or unstable system behavior. The need for resilient observers that can resist cyberattacks has long been recognized [1]. Resilient observer design yields observers that can continue to function properly and yield reliable estimates in spite of implementation errors or cyberattacks [2]–[3][4][5].

A major work on resilience of observers in the face of cyberattacks was the paper by Fawzi et al. [2]. The authors showed that it is impossible to reconstruct the state of the system if more than half of the sensors are attacked. Yaz et al. presented a simple resilient observer design using linear matrix inequalities (LMIs) [3]. They obtained results for bounded estimation error, an H2 observer and a strictly input passive observer. The authors also investigate the dissipativity properties of their design by appropriate choice of supply rate. In [4], the authors designed a resilient observer in the presence of noise and modeling errors using LMIs. In [5], the authors present an efficient resilient observer with complexity $O(np)$, where n is the order of the system and p is the number of sensors.

The designs of [3] and [4] are feasible and simple because of the powerful theory available for solving LMIs [6]–[7][8][9]. This paper obtains new LMIs whose solution provides a resilient observer for a linear discrete-time system subject to cyberattacks or failure. The observer is designed to be Q-S-R dissipative [10]. The choice of design parameters allows us to design a passive observer, a strictly input passive observer, a strictly output passive observer, or a very strictly passive observer. The design is resilient with respect to errors in observer implementation, as well as to disturbance inputs. The disturbance input can represent malicious control signal sent by an attacker. We assume that the observer gain errors are bounded with a known bound. Three examples are provided to demonstrate the simplicity and effectiveness of the design approach. The first example considers the disturbance-free case, the second considers the observer in the presence of an observer, and the third is a population model for the female of a species. The population model example considers the effect of environmental hazards on the species and the observer serves to obtain estimates of three age groups of the female population subject to environmental hazards.

The paper is organized as follow. Section II provides the plan model used in the paper and the corresponding observer. It also reviews the definition of dissipativity that is used in the design procedure. Section III presents our observer design including three examples, and Section IV is the conclusion.

SECTION II. Model and Performance Criteria

Consider the discrete time system

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k + Fw_k \quad (1)$$

with the measurement equation

$$\boldsymbol{y}_k = C\mathbf{x}_k + D\mathbf{u}_k + G\boldsymbol{w}_k \quad (2)$$

where $\boldsymbol{x}_k \in \mathcal{R}^n$is the state vector, $\boldsymbol{u}_k \in \mathcal{R}^p$is the control input, $\boldsymbol{w}_k \in \mathcal{R}^p$is a disturbance input, and $\boldsymbol{y}_k \in \mathcal{R}^p$is the measurement vector. We design a state estimator for the system, with gain L, that is robust with respect to errors in the observer implementation. The estimator is of the form

$$\hat{\boldsymbol{x}}_{k+1} = A\hat{\boldsymbol{x}}_k + B\boldsymbol{u}_k + (L + \Delta L)[\boldsymbol{y}_k - C\hat{\boldsymbol{x}}_k - Du_k] \quad (3)$$

The observer gain error ΔL is subject to the perturbation bound

$$\Delta L \Delta L^T \leq \gamma I_p \quad (4)$$

The estimation error $\boldsymbol{e}_k = x_k - \widehat{\boldsymbol{x}}_k$ is governed by the dynamic equation

$$\begin{aligned} \boldsymbol{e}_{k+1} &= \tilde{A}\boldsymbol{e}_k + \tilde{F}\boldsymbol{w}_k \\ \tilde{A} &= A - (L + \Delta L)C = \overline{A} - \Delta LC \\ \tilde{F} &= F - (L + \Delta L)G = \overline{F} - \Delta LG \\ \overline{A} &= A - LC, \overline{F} = F - LG \end{aligned} \quad (5)(6)$$

The performance output is

$$\begin{aligned} \boldsymbol{z}_k &= C_z\boldsymbol{e}_k + D_z\boldsymbol{w}_k \\ \boldsymbol{z}_k &\in \mathcal{R}^p \end{aligned} \quad (7)$$

The solution of the error dynamics equation is

$$\boldsymbol{e}_k = \tilde{A}^k\boldsymbol{e}_0 + \sum_{i=0}^{k-1} \tilde{A}^{k-i-1}\tilde{F}\boldsymbol{w}_i \quad (8)$$

The error converges to zero if the eigenvalues of the perturbed matrix $\tilde{A}$ remain inside the unit circle for any perturbation from the nominal gain bounded as in **(4).** The nominal gain L is chosen so that all the eigenvalues of the matrix $\tilde{A}$ are inside the unit circle. We require the following property for the error dynamics.

Definition 1

SECTION Definition 1

Dissipativity

The error dynamics (5–7) is dissipative with respect to a supply rate $W(\boldsymbol{u}_k, \boldsymbol{z}_k)$ if there exists a nonnegative storage function $V: \mathcal{R}^n \to \mathcal{R}$ such that for all $\boldsymbol{u}_k \in \boldsymbol{\mathcal{R}}^p$and all k

$$V_{k+1} - V_k \leq w(\boldsymbol{u}_k, \boldsymbol{z}_k) \quad (9)$$

If the supply rate has the form $\boldsymbol{w}(\boldsymbol{u}_k, \boldsymbol{z}_k) = \boldsymbol{u}_k^T\boldsymbol{z}_k$, then the system is called passive. A common choice of the supply rate is the quadratic

$$w(\boldsymbol{u}_k, \boldsymbol{z}_k) = \boldsymbol{z}_k^TQ\boldsymbol{z}_k + \boldsymbol{w}_k^TR\boldsymbol{w}_k + z_k^TS\boldsymbol{w}_k \quad (10)$$

with symmetric matrices Q and R. If a system has this property then it is called QSR dissipative [7]. One choice of the matrices is [10]

$$Q = -\delta\boldsymbol{I}_p, \boldsymbol{R} = -\epsilon\boldsymbol{I}_p, \boldsymbol{S} = \beta\boldsymbol{I}_p, \delta, \boldsymbol{\epsilon}, \beta \in \boldsymbol{\mathcal{R}} \quad (11)$$

Table I. Dissipativity and choice of Q, S, R, MATRICES WITH $\delta, \epsilon, \beta > 0$

Name	Q, S, R	Inequality
Passive	$Q = 0, R = 0, S = 0.5I_p$	$\boldsymbol{u}_k^T\boldsymbol{z}_k \geq V_{k+1} - V_k$
Strictly Passive	$Q = 0, R = -\epsilon I_p, S = 0.5I_p$	$\boldsymbol{u}_k^T\boldsymbol{z}_k - \epsilon\boldsymbol{w}_k^T\boldsymbol{w}_k \geq V_{k+1} - V_k$
Strictly Output Passive	$Q = -\delta I_p, R = 0, S = 0.5I_p$	$\boldsymbol{u}_k^T\boldsymbol{z}_k - \delta\boldsymbol{z}_k^T\boldsymbol{z}_k \geq V_{k+1} - V_k$
Very Strictly Passive	$Q = -\delta I_p, R = -\epsilon I_p, S = 0.5I_p$	$\boldsymbol{u}_k^T\boldsymbol{z}_k - \delta\boldsymbol{z}_k^T\boldsymbol{z}_k - \epsilon\boldsymbol{w}_k^T\boldsymbol{w}_k \geq V_{k+1} - V_k$

SECTION III. Observer Design

We define a quadratic error energy function whose decay is a measure of the performance of the state estimator

$$V_k = \boldsymbol{e}_k^T P \boldsymbol{e}_k, \ P > 0 \ (12)$$

We require the error energy function and the error supply rate to satisfy

$$\boldsymbol{z}_k^T Q \boldsymbol{z}_k + \boldsymbol{w}_k^T R \boldsymbol{w}_k + \boldsymbol{z}_k^T S \boldsymbol{w}_k \geq V_{k+1} - V_k \ (13)$$

Substituting for the error energy and for the performance output gives

$$\begin{aligned} [C_z\boldsymbol{e}_k + D_z\boldsymbol{w}_k]^T \quad & Q[C_z\boldsymbol{e}_k + D_z\boldsymbol{w}_k] + \boldsymbol{w}_k^T R \boldsymbol{w}_k \\ & +[C_z\boldsymbol{e}_k + D_z\boldsymbol{w}_k]^T S \boldsymbol{w}_k \\ \geq \boldsymbol{e}_k^T[\tilde{A}^T P \tilde{A} - P] \quad & \boldsymbol{e}_k + \boldsymbol{w}_k^T \tilde{F}^T P \tilde{F} \boldsymbol{w}_k + 2\boldsymbol{w}_k^T \tilde{F}^T P \tilde{A} \boldsymbol{e}_k \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & -\boldsymbol{w}_k^T[\tilde{F}^T P \tilde{F} - D_z^T S - R - D_z^T Q D_z]\boldsymbol{w}_k \\ & \geq \boldsymbol{e}_k^T[\tilde{A}^T P \tilde{A} - P - C_z^T Q C_z]\boldsymbol{e}_k \qquad (14) \\ & +\boldsymbol{w}_k^T[2\tilde{F}^T P \tilde{A} - S C_z - 2D_z^T Q C_z]\boldsymbol{e}_k \end{aligned}$$

We rewrite the inequality as the matrix inequality

$$\begin{bmatrix} P + C_z^T Q C_z & C_z^T Q D_z + \frac{C_z^T S}{2} \\ D_z^T Q C_z + \frac{S C_z}{2} & \frac{D_z^T S + S D_z}{2} + R + D_z^T Q D_z \end{bmatrix} \ (15)$$
$$- \begin{bmatrix} \tilde{A}^T P \tilde{A} & \tilde{A}^T P \tilde{F} \\ \tilde{F}^T P \tilde{A} & \tilde{F}^T P \tilde{F} \end{bmatrix} - \begin{bmatrix} \tilde{A}^T P \tilde{A} & \tilde{A}^T P \tilde{F} \\ \tilde{F}^T P \tilde{A} & \tilde{F}^T P \tilde{F} \end{bmatrix} \geq 0$$

Case I: No Disturbance

In the absence of a disturbance we have

$$P + C_z^T Q C_z - \tilde{A}^T P \tilde{A} \geq 0 \ (16)$$

Since Q is negative definite, condition **(16)** ensures the asymptotic stability of the perturbed observer dynamics. With $Q = -\delta I$, and with the pair $((\tilde{A}, C_z))$ observable, the constant δ determines the excess passivity of the observer. If the constant is negative, then it indicates the passivity deficiency.

In the absence of a disturbance, use Schur's complement (see Lemma 1 in the Appendix) to obtain

$$\begin{bmatrix} P + C_z^T Q C_z & \tilde{A}^T P \\ P\tilde{A} & P \end{bmatrix} \geq 0 \ (17)$$

Substituting for $\tilde{A}$ gives

$$\begin{bmatrix} P + C_z^T Q C_z & (PA - YC - P\Delta LC)^T \\ PA - YC - P\Delta LC & P \end{bmatrix} \geq 0 \ (18)$$

With no perturbation in the observer gain, i.e. $\Delta L = 0$, applying the Schur complement to the nominal system gives the condition

$$\begin{bmatrix} P + C_z^T Q C_z & (PA - YC)^T \\ PA - YC & P \end{bmatrix} \geq 0 \ (19)$$

The last inequality can be solved for P and Y and the result is used to obtain $L = P^{-1}Y$. With $C_z = 0$and with strict inequality, the above condition implies that the pair (A, C)must be detectable.

For the perturbed case, we use Lemma 2 (see the Appendix) to write

$$\begin{bmatrix} \alpha C^T C & 0 \\ 0 & \alpha^{-1}\gamma P^2 \end{bmatrix} \geq \begin{bmatrix} 0 & -C^T P \Delta L \\ -\Delta L^T P C & 0 \end{bmatrix} \ (20)$$

This yields he sufficient condition for the inequality (18)

$$\begin{bmatrix} P + C_z^T Q C_z - \alpha C^T C & (PA - YC)^T \\ PA - YC & P - \alpha^{-1}\gamma P^2 \end{bmatrix} \geq 0 \ (21)$$

Using Schur's complement gives

$$\begin{bmatrix} P + C_z^T Q C_z - \alpha C^T C & (PA - YC)^T & 0 \\ PA - YC & P & P \\ 0 & P & \alpha^{-1}\gamma I_n \end{bmatrix} \geq 0 \ (22)$$

If the uncertainty bound γ is unknown, we rewrite (22) as

$$\begin{bmatrix} P + C_z^T Q C_z - \alpha C^T C & (PA - YC)^T & 0 \\ PA - YC & P & P \\ 0 & P & \beta I_n \end{bmatrix} \geq 0 \ (23)$$

with $\beta = \alpha^{-1}\gamma$and solve the LMI for P, Y, α, β. This gives a bound $\gamma = \alpha\beta$

If the uncertainty bound γ is known, we can set $\alpha = \gamma$, to obtain the simpler form

$$\begin{bmatrix} P + C_z^T Q C_z - \gamma C^T C & (PA - YC)^T & 0 \\ PA - YC & P & P \\ 0 & P & I_n \end{bmatrix} \geq 0 \ (24)$$

and solve the LMI for P, Y, α

Example 1

Consider the system with

$$A = \begin{bmatrix} 0 & 1 \\ -0.3 & 0.2 \end{bmatrix}, C = [11], D = 0$$
$$C_z = [10], D_z = 02, \gamma = 0.1$$

We first set $Q = -0.1$and use the LMI Toolbox of MATLAB to solve the LMI (24). We obtain the observer gain

$$L = [4.4772 \quad -0.7620]^T$$

The observer matrix is stable with the eigenvalues

$$\{-0.1719, 0.0003\}$$

The parameter $\alpha = 0.1814$ and the matrix

$$P = \begin{bmatrix} 28.3857 & -8.4749 \\ -8.4749 & 38.9697 \end{bmatrix}$$

is clearly positive definite.

For an observer with greater excess passivity, we use $Q = -1.0$ and this gives the eigenvalues

$$\{-0.1682, 0.0017\}$$

with the gain

$$L = [4.4450 \quad -0.7804]^T$$

The parameter $\alpha = 25.1280$ and the matrix

$$P = \begin{bmatrix} 160.5207 & -55.6979 \\ -55.6979 & 216.0581 \end{bmatrix}$$

is clearly positive definite.

In the disturbance-free case with an unknown uncertainty bound γ, we solve (23) with $\alpha = 3.0193, \gamma = 0.0779$. From earlier calculations, we know that we can design a passive observer with $\gamma = 0.1$. This shows that the bound obtained using (23) is not the lowest admissible bound. For this feasible but conservative solution, we have the eigenvalues

$$\{-0.1606, -0.0082\}$$

with the gain

$$L = [0.4450 \quad -0.0762]^T$$

The matrix

$$P = \begin{bmatrix} 13.1335 & -2.5253 \\ -2.5253 & 17.9593 \end{bmatrix}$$

The matrix is clearly positive definite.

Case II: Disturbance

We first rewrite the inequality (15) as

$$\begin{aligned} & M - \begin{bmatrix} \tilde{A}^T \\ \tilde{F}^T \end{bmatrix} P \begin{bmatrix} \tilde{A} \tilde{F} \end{bmatrix} \geq 0 \\ & M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \\ & = \begin{bmatrix} P + C_z^T Q C_z & C_z^T Q D_z + \frac{C_z^T S}{2} \\ D_z^T Q C_z + \frac{S C_z^T}{2} & \frac{D_z^T S + S D_z}{2} + R + D_z^T Q D_z \end{bmatrix} \end{aligned} \quad (25)(26)$$

Substituting for $\tilde{A}$ and $\tilde{F}$, then using Schur's complement, we have

$$\begin{aligned} & \begin{bmatrix} M & M_a^T \\ M_a & P \end{bmatrix} \geq 0 \\ & M_a = [PA - YC - P\Delta LC \quad PF - YG - P\Delta LG] \\ & = [PA - YC \quad PF - YG] - P\Delta L G_c \\ & G_c = [C \quad G] \end{aligned} \quad (27)(28)$$

From Lemma 2, we write

$$\begin{bmatrix} \alpha G_c^T G_c & 0 \\ 0 & \alpha^{-1}\gamma P^2 \end{bmatrix} \geq \begin{bmatrix} 0 & -G_c^T \Delta L^T P \\ -P\Delta L G_c & 0 \end{bmatrix} \quad (29)$$

Using similar steps to the disturbance-free case

$$\begin{array}{lll} M & [PA - YCPF - YG]^T & 0 \\ [[PA - YCPF - YG] & P & P\ ] \\ 0 & P & \beta I_n \end{array} \quad (30)$$
$$\geq 0$$
$$\beta = \alpha^{-1}\gamma$$

In summary, to design the observer we need to solve the LMI (30) using the MATLAB LMI toolbox for Y, P, β, *or* α.

Example 2

Consider the system of Example 1 with the matrices

$$F = [1\ 1]^T, G = 1$$

We use a tighter bound on the gain perturbation

$$\gamma = 0.05,$$

For a passive observer, we use $Q = 0, R = 0, S = 0.5$, we get $\alpha = 0.\ 0777$ and the observer gain

$$L = [0.2510 - 0.1922]^T$$

The corresponding eigenvalues are

$$\{-0.0799, 0.2211\}$$

with the positive definite matrix

$$P = \begin{bmatrix} 0.4359 & -0.4033 \\ -0.403 & 0.9102 \end{bmatrix}$$

For a strictly output passive observer, we use $Q = -0.01, R = 0, S = 0.5$. We obtain $\alpha = 0.0738$and the observer gain

$$L = [0.2565 \quad -0.1887]^T$$

The corresponding eigenvalues are

$$\{-0.0798, 0.2120\}$$

with the positive definite matrix

$$P = \begin{bmatrix} 0.4187 & -0.3833 \\ -0.3833 & 0.8677 \end{bmatrix}$$

For a strictly input passive observer, we use $Q = 0.0, R = -0.01, S = 0.5$. This gives $a = 0.0772$and the observer gain

$$L = [0.2492 \quad -0.1933]^T$$

The corresponding eigenvalues are

$$\{-0.08, 0.2241\}$$

with the positive definite matrix

$$P = \begin{bmatrix} 0.4336 & -0.4017 \\ -0.4017 & 0.6056 \end{bmatrix}$$

For a very strictly passive observer, we use

$$Q = -0.01, R = -0.01, S = 0.5, n = 0.0732$$

and obtain the observer gain

$$L = [0.2544 \quad -0.1900]^T$$

The corresponding eigenvalues are

$$\{-0.0799, 02155\}$$

with the positive definite matrix

$$P = \begin{bmatrix} 0.4160 & -0.3811 \\ -0.3811 & 0.8621 \end{bmatrix}$$

In the presence of a disturbance with an unknown uncertainty bound γ, we solve (30) with $Q = -0.1, S = 0.5$, to obtain $\alpha = 0.0093, \gamma = 0.0084$. From earlier calculations, we know that we can design a passive observer with $\gamma = 0.05$. This shows that the bound obtained using (30) is not the lowest admissible bound. For this feasible but conservative solution, we have the eigenvalues

$$\{-0.0882, 0.3766\}$$

with the gain

$$\begin{aligned} L &= [0.1632 \quad -0.2516]^T \\ P &= \begin{bmatrix} 0.2603 & -0.1583 \\ -0.1583 & 0.4374 \end{bmatrix} \end{aligned}$$

The matrix is positive definite.

Example 3

Leslie Age-structured Population Model

We present a model for the female population of a species with a maximum age of 3 years based on an example from [11]. The population is divided into three groups according to age; namely the age groups (0,1), [1,2), [2], [3]. These three populations are state variables of the model $\{x_{1k}, x_{2k}, x_{3k}\}$, respectively. The second and third populations can produce offspring but the second population is more fertile.

To preserve the species, only the mature population x_3 is harvested. However, all three populations can be affected by adverse environmental conditions with the first population being the most affected. These adverse factors are represented by a scalar disturbance input w_k. The state equation for this model is given by

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 6 & \frac{10}{3} \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k + \begin{bmatrix} 1 \\ 0.1 \\ 0.05 \end{bmatrix} w$$

Because of the difficulty of estimating the age of members of the species, the only measurement available is the total population and the measurement equation is

$$y_k = [1 \quad 1 \quad 1]x$$

For conservation purposes, the second is the main variable of interest as governed by the equation

$$z_k = [0 \quad 1 \quad 0]\boldsymbol{x}_k + 2w_k$$

For the disturbance-free case we use $\gamma = 0.01, Q = -0.1$, and obtain $\alpha = 0.1887$ The observer gain matrix is

$$L = [0.40 \quad 0.6064 \quad -0.0043]^T$$

and the corresponding eigenvalues are

$$\{-0.0143, -0.3009 \pm \mathrm{j}0.3708\}$$

are inside the unit circle. The matrix

$$P = \begin{bmatrix} 0.6276 & 0.7132 & 0.5386 \\ 0.7132 & 7.6632 & 2.9555 \\ 0.5386 & 2.9555 & 9.0062 \end{bmatrix}$$

In the presence of a disturbance input we use

$$\gamma = 0.001, Q = -0.001, R = S = 01$$
$$L = [0.0300 \quad 0.4976 \quad 0.0336]^T$$

and the corresponding eigenvalues are

$$\{-0.1411, -0.3527 \pm \mathrm{j}0.5535\}$$

are inside the unit circle. The matrix

$$P = \begin{bmatrix} 4.7768 & 1.1001 & 1.2087 \\ 1.1001 & 58.6252 & 16.4968 \\ 1.2087 & 16.4968 & 89.9181 \end{bmatrix} \times 10^{-2}$$

The constant α is positive but small with

$$\alpha = 2.2258 \times 10^{-3}$$

SECTION IV. Conclusion

This paper introduces a new resilient observer design that dissipates the error energy to converge to the correct state estimates to provide immunity from cyberattacks and implementation errors. By choosing the weight matrices in the dissipativity inequality, the designer can change the excess passivity of the observer and the level of tolerance to implementation errors and cyberattacks. By appropriate choice of weight matrices, the designer can obtain a passive observer, a strictly input passive observer, a strictly output passive observer, or a very strictly passive observer. The observer can be designed both in the case where a bound in the gain perturbation is known and the case where the bound is unknown. The LMIs used in the observer design can yield an estimate of the allowable observer gain perturbation. However, the corresponding perturbation bound only provides a sufficient condition and viable observers can be obtained with a less restrictive perturbation bound. In general, the presence of a disturbance necessitates using a more restrictive bound on the allowable gain error for an acceptable observer design.

Appendix

Lemma 1:

Lemma 1: The Schur Complement [6]

For matrices $Q = Q^T, S = S^T$, then for any compatible matrix R

$$\begin{bmatrix} Q & R \\ R^T & S \end{bmatrix} > 0 \quad \Leftrightarrow S > 0, Q - RS^{-1}R^T > 0$$
$$\Leftrightarrow Q > 0, S - R^T Q^{-1} R > 0$$

Lemma 2

$$\exists \alpha > 0, \begin{bmatrix} \alpha Q_1^T Q_1 & \pm Q_1^T Q_2 \\ \pm Q_2^T Q_1 & \alpha^{-1} Q_2^T Q_2 \end{bmatrix} \geq 0$$

In addition, if $Q_1^T Q_1 < \gamma Q_4$, then

$$\exists \alpha > 0, \begin{bmatrix} \alpha \gamma Q_4 & \pm Q_1^T Q_2 \\ \pm Q_2^T Q_1 & \alpha^{-1} Q_2^T Q_2 \end{bmatrix} \geq 0$$

Proof

We write the inequality as a quadratic form

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \alpha Q_1^T Q_1 & \pm Q_1^T Q_2 \\ \pm Q_2^T Q_1 & \alpha^{-1} Q_2^T Q_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq 0$$

We expand the quadratic form as

$$\alpha \boldsymbol{x}^T Q_1^T Q_1 \boldsymbol{x} + \alpha^{-1} Q^T Q_2 \boldsymbol{y} \pm 2\boldsymbol{x}^T Q_1^T Q_2 \boldsymbol{y} \geq 0$$

The quadratic form can be rewritten as

$$(\sqrt{\alpha} Q_1 \boldsymbol{x} \pm Q_2 \frac{\boldsymbol{y}}{\sqrt{\alpha}})^T (\sqrt{\alpha} Q_1 \boldsymbol{x} \pm Q_2 \frac{\boldsymbol{y}}{\sqrt{\alpha}}) \geq 0$$

If $Q_1^T Q_1 < \gamma Q_4$, then

$$\alpha \gamma \boldsymbol{x}^T Q_4 \boldsymbol{x} + \alpha^{-1} Q_2^T Q_2 \boldsymbol{y} \pm 2\boldsymbol{x}^T Q_1^T Q_2 \boldsymbol{y}$$
$$\geq \alpha \boldsymbol{x}^T Q_1^T Q_1 \boldsymbol{x} + \alpha^{-1} Q_2^T Q_2 \boldsymbol{y} \pm 2\boldsymbol{x}^T Q_1^T Q_2 \boldsymbol{y} \geq 0$$

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Keywords

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