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H_2−H_∞ control of discrete-time nonlinear systems using the state-dependent Riccati equation approach

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ABSTRACT

A novel H_2−H_∞ State-dependent Riccati equation control approach is presented for providing a generalized control framework to discrete-time nonlinear system. By solving a generalized Riccati equation at each time step, the nonlinear state feedback control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_∞ type of disturbance attenuation. Two numerical techniques to compute the solution of the resulting Riccati equation are presented: The first one is based on finding the steady-state solution of the difference equation at every step and the second one is based on finding the minimum solution of a linear matrix inequality. The effectiveness of the proposed techniques is demonstrated by simulations involving the control of an inverted pendulum on a cart, a benchmark mechanical system.

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Introduction

The Hamilton–Jacobi equation (HJE) is a traditional approach to characterize the optimal control of nonlinear systems. The solution of the HJEs provides the necessary and sufficient optimal control conditions for system modelled by nonlinear dynamics. When the controlled system is linear time-invariant and the performance index is linear quadratic regulator (LQR), the HJEs can be reduced to algebraic Riccati equations (AREs). As for H_∞ nonlinear control problem, the optimal control solution is equivalent to solving the corresponding Hamilton–Jacobi inequalities (HJIs). However, HJEs and HJIs, which are first-order partial differential equations and inequalities, cannot be solved for more than a few state variables.

Motivated by the success of linear system optimal control methods, there has been a great deal of research involves in approximating the solutions of HJEs and HJIs over the last decade. As powerful alternatives to HJE/HJI techniques: the state-dependent linear matrix inequality (SDLMI) and the state-dependent Riccati equation (SDRE) techniques have provided us very effective algorithms for synthesizing the nonlinear feedback controls. Both SDLMI and SDRE utilize state-dependent linear representations, some of the earliest work can be found in Cloutier (1997), Cloutier, D'Souza, and Mracek (1996); Huang and Lu (1996) and Mohseni, Yaz, and Olejniczak (1998).

The purpose behind SDLMI is to convert a nonlinear system control design into a convex optimization problem involving state-dependent linear matrix inequality solutions. The recent development in numerical algorithms for solving convex optimization provides very efficient means for solving LMI (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). If a solution can be expressed in LMI form, then there exist efficient algorithms providing globally optimal numerical solutions. Therefore, if the LMIs are feasible, then SDLMI control technique provides optimal solutions at each step for a given state for nonlinear system control problems. As pointed out in Jeong, Feng, Yaz, and Yaz (2010), Wang and Yaz (2009), Wang, Yaz, and Jeong (2010) and Wang, Yaz, and Yaz (2010), SDLMI provides us an effective method to synthesize nonlinear feedback control in achieving nonlinear quadratic regulator (NLQR), H_∞ and positive realness performance criteria.

The SDRE control has emerged as general design method since the mid-1990s, which provides a systematic and effective design framework for nonlinear systems. Motivated by linear quadratic regulator control by algebraic Riccati equation (ARE), Cloutier et al. extended the

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result to nonlinear quadratic regulator problem by using state-dependent coefficient matrices as pointed out in Cloutier (1997) and Cloutier et al. (1996). A discrete SDRE method is developed in Dutka, Ordys, and Grimble (2005). Due to the computational advantage and guaranteed local stability, the SDRE method is of practical importance and has a wide range of applications, including robotics, missiles, aircraft, satellite/spacecraft, unmanned aerial vehicles (UAVs), ship systems, autonomous underwater vehicles, automotives, process control, chaotic systems, biomedical systems, guidance and navigation, etc. A recent survey of the development of SDRE method can be found in Cimen (2008, 2010).

Traditionally, the SDRE method approaches address the nonlinear quadratic regulator problem. The contribution of this manuscript is to propose a novel H_2−H_∞ SDRE control approach with the purpose of providing a generalized control framework to discrete-time nonlinear systems. By solving the generalized SDRE at each time step, the optimal control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_∞ type of disturbance reduction (Basar & Bernhard, 1995; Van der Shaft, 1993). Two numerical solution procedures: one involving the steady-state solution of a generalized Riccati difference equation and the other involving a state-dependent LMI are also given. The effectiveness of the proposed technique is demonstrated by simulations involving the control of a benchmark mechanical system.

The paper is organized as follows: In the second section, the system model and the performance criteria are introduced. In the third section, the derivation of the H_2−H_∞ SDRE controller is provided. Optimal control solution can be obtained by solving the generalized SDRE. To solve the generalized SDRE, a difference SDRE and an SDLMI solution are also presented to provide computational alternatives. The fourth section contains an illustrative example involving the control of the inverted pendulum on a cart. Finally, the conclusions are summarized in the fifth section. The following notation is used in this work: $x \in \Re^n$ denotes n-dimensional real vector with norm $\|x\| = (x^{\mathrm{T}}x)^{1/2}$ where $(\cdot)^{\mathrm{T}}$ indicates transpose. $A \geq 0$ for a symmetric matrix denotes a positive semi-definite matrix. l_2 is the space of infinite sequences of finite-dimensional vectors with finite energy: $\sum_{k=0}^{\infty} \|x_k\|^2 < \infty$.

System model and performance index

Consider the input affine discrete-time nonlinear system given by the following difference equation:

$$\begin{aligned} x_{k+1} &= A(x_k)x_k + B(x_k)u_k + F(x_k)w_k \\ &= A_k x_k + B_k u_k + F_k w_k, \end{aligned} \tag{1}$$

where $x_k \in \Re^n$ is the state vector, $u_k \in \Re^m$ the applied input, $w_k \in \Re^q$ the l_2 type of disturbance and A_k, B_k, F_k the state-dependent matrices of known structure.

Note that the simplified notation for time-varying matrices A_k, B_k, etc. is used to denote the state-dependent matrices. The performance output function $z_k \in \Re^p$ is generalized as follows:

$$\begin{aligned} z_k &= C(x_k)x_k + D(x_k)u_k + G(x_k)w_k \\ &= C_k x_k + D_k u_k + G_k w_k, \end{aligned} \tag{2}$$

where C_k, D_k, G_k are state-dependent coefficient matrices of known structure.

It is assumed that the state feedback is available. Otherwise, estimated state variable can be obtained from a nonlinear state estimator. The nonlinear state feedback control input is given by

$$u_k = K(x_k) \cdot x_k = K_k \cdot x_k. \tag{3}$$

Consider the quadratic energy function

$$V_k = x_k^{\mathrm{T}} P_k x_k > 0 \tag{4}$$

for the following difference inequality:

$$V_{k+1} - V_k + x_k^{\mathrm{T}} Q_k x_k + u_k^{\mathrm{T}} R_k u_k + z_k^{\mathrm{T}} z_k - \gamma^2 w_k^{\mathrm{T}} w_k \leq 0 \tag{5}$$

with $Q_k > 0, R_k > 0$ being the function of x_k.

Note that upon summation over k, Equation (5) yields

$$\begin{aligned} V_N + \sum_{k=0}^{N-1} [x_k^{\mathrm{T}} Q_k x_k + u_k^{\mathrm{T}} R_k u_k + \|z_k\|^2] \leq V_0 \\ + \gamma^2 \sum\nolimits_{k=0}^{N-1} [\|w_k\|^2]. \end{aligned} \tag{6}$$

Notice that Q_k and R_k are state-dependent counter parts of the weighting matrices in the traditional linear quadratic (H_2) control approach and γ^2 is the H_∞ bound. By properly specifying the value of the weighing matrices Q_k, R_k, C_k, D_k, mixed performance criteria can be used in nonlinear control design, which yields a mixed NLQR in combination with H_∞ performance index.

Main results

The following theorem summarizes the main results of the paper:

Theorem 1: *Given the system* (1), *performance output* (2), *and control input* (3), *the mixed performance index* (6) *can* *be achieved by using the control feedback*

$$K_k^o = -\left\{ \begin{array}{c} R_k + B_k^\mathrm{T} P_k B_k + D_k^\mathrm{T} D_k - \\ \begin{pmatrix} B_k^\mathrm{T} P_k F_k \\ +D_k^\mathrm{T} G_k \end{pmatrix} \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{pmatrix} B_k^\mathrm{T} P_k F_k \\ +D_k^\mathrm{T} G_k \end{pmatrix}^\mathrm{T} \end{array} \right\}^{-1} \times \left\{ \begin{array}{c} B_k^\mathrm{T} P_k A_k + D_k^\mathrm{T} C_k - \\ \begin{pmatrix} B_k^\mathrm{T} P_k F_k + \\ D_k^\mathrm{T} G_k \end{pmatrix} \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{pmatrix} A_k^\mathrm{T} P_k F_k + \\ C_k^\mathrm{T} G_k \end{pmatrix}^\mathrm{T} \end{array} \right\}, \tag{7}$$

where P_k is obtained from the generalized SDRE:

$$\begin{aligned} P_k = &\left\{ A_k^\mathrm{T} P_k A_k + C_k^\mathrm{T} C_k + Q_k - \begin{bmatrix} A_k^\mathrm{T} P_k F_k \\ +C_k^\mathrm{T} G_k \end{bmatrix} \cdot \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} A_k^\mathrm{T} P_k F_k \\ +C_k^\mathrm{T} G_k \end{bmatrix}^\mathrm{T} \right\} \\ &- \left\{ A_k^\mathrm{T} P_k B_k + C_k^\mathrm{T} D_k - (A_k^\mathrm{T} P_k F_k + C_k^\mathrm{T} G_k) \times \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} (B_k^\mathrm{T} P_k F_k + D_k^\mathrm{T} G_k)^\mathrm{T} \right\} \\ &\times \left\{ R_k + B_k^\mathrm{T} P_k B_k + D_k^\mathrm{T} D_k - \begin{pmatrix} B_k^\mathrm{T} P_k F_k \\ +D_k^\mathrm{T} G_k \end{pmatrix} \times \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{pmatrix} B_k^\mathrm{T} P_k F_k \\ +D_k^\mathrm{T} G_k \end{pmatrix}^\mathrm{T} \right\}^{-1} \\ &\times \left\{ A_k^\mathrm{T} P_k B_k + C_k^\mathrm{T} D_k - (A_k^\mathrm{T} P_k F_k + C_k^\mathrm{T} G_k) \times \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} (B_k^\mathrm{T} P_k F_k + D_k^\mathrm{T} G_k)^\mathrm{T} \right\}. \end{aligned} \tag{8}$$

Proof: By applying system (1), performance output (2), control input (3), performance index (5) can be written as

$$\begin{aligned} &[(A_k + B_k K_k) x_k + F_k w_k]^\mathrm{T} P_{k+1} [(A_k + B_k K_k) x_k + F_k w_k] \\ &\quad - x_k^\mathrm{T} P_k x_k + x_k^\mathrm{T} Q x_k + u_k^\mathrm{T} R u_k + [C_k x_k + D_k u_k + G_k w_k]^\mathrm{T} \\ &\quad \times [C_k x_k + D_k u_k + G_k w_k] - \gamma^2 w_k^\mathrm{T} w_k \le 0. \end{aligned} \tag{9}$$

Equivalently, we have

$$[x_k^\mathrm{T} \quad w_k^\mathrm{T}] \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \le 0, \tag{10}$$

$$\begin{aligned} \Xi_{11} &= (A_k + B_k K_k)^\mathrm{T} P_{k+1} (A_k + B_k K_k) - P_k + Q_k \\ &\quad + K_k^\mathrm{T} R_k K_k + (C_k + D_k K_k)^\mathrm{T} (C_k + D_k K_k), \\ \Xi_{12} &= (A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k + (C_k + D_k K_k)^\mathrm{T} G_k, \\ \Xi_{22} &= F_k^\mathrm{T} P_{k+1} F_k + G_k^\mathrm{T} G_k - \gamma^2 I. \end{aligned} \tag{11}$$

Therefore, we have

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & \Delta_{22} \end{bmatrix} \ge 0, \tag{12}$$

where

$$\begin{aligned} \Delta_{11} &= P_k - [(A_k + B_k K_k)^\mathrm{T} P_{k+1} (A_k + B_k K_k) + Q_k \\ &\quad + K_k^\mathrm{T} R_k K_k + (C_k + D_k K_k)^\mathrm{T} (C_k + D_k K_k)], \\ \Delta_{12} &= -[(A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k + (C_k + D_k K_k)^\mathrm{T} G_k], \\ \Delta_{22} &= -[F_k^\mathrm{T} P_{k+1} F_k + G_k^\mathrm{T} G_k - \gamma^2 I]. \end{aligned} \tag{13}$$

By applying the Schur complement (Boyd et al., 1994), we obtain

$$\begin{aligned} &P_k - [(A_k + B_k K_k)^\mathrm{T} P_{k+1} (A_k + B_k K_k) + Q_k + K_k^\mathrm{T} R_k K_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} (C_k + D_k K_k)] \\ &\quad + [(A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k + (C_k + D_k K_k)^\mathrm{T} G_k] \\ &\quad \times [F_k^\mathrm{T} P_{k+1} F_k + G_k^\mathrm{T} G_k - \gamma^2 I]^{-1} [(A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} G_k]^\mathrm{T} \ge 0, \end{aligned} \tag{14}$$

which yields

$$\begin{aligned} P_k &\ge [(A_k + B_k K_k)^\mathrm{T} P_{k+1} (A_k + B_k K_k) + Q_k + K_k^\mathrm{T} R_k K_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} (C_k + D_k K_k)] - [(A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} G_k] \times [F_k^\mathrm{T} P_{k+1} F_k + G_k^\mathrm{T} G_k - \gamma^2 I]^{-1} \\ &\quad \times [(A_k + B_k K_k)^\mathrm{T} P_{k+1} F_k + (C_k + D_k K_k)^\mathrm{T} G_k]^\mathrm{T}. \end{aligned} \tag{15}$$

The minimum value of P_k is achieved when the inequality above is satisfied as an equality. Since the iterative solution starts at P_∞ and runs backward in time and for $P_{k+1} = P_k$ convergence occurs, the difference equation becomes an algebraic equation (Dutka et al., 2005) as follows:

$$\begin{aligned} P_k &= [(A_k + B_k K_k)^\mathrm{T} P_k (A_k + B_k K_k) + Q_k + K_k^\mathrm{T} R_k K_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} (C_k + D_k K_k)] - [(A_k + B_k K_k)^\mathrm{T} P_k F_k \\ &\quad + (C_k + D_k K_k)^\mathrm{T} G_k] \times [F_k^\mathrm{T} P_k F_k + G_k^\mathrm{T} G_k - \gamma^2 I]^{-1} \\ &\quad \times [(A_k + B_k K_k)^\mathrm{T} P_k F_k + (C_k + D_k K_k)^\mathrm{T} G_k]^\mathrm{T}. \end{aligned} \tag{16}$$

By collecting terms, we have

$$P_k = \left\{ A_k^\mathrm{T} P_k A_k + C_k^\mathrm{T} C_k + Q_k - [A_k^\mathrm{T} P_k F_k + C_k^\mathrm{T} G_k] \cdot \begin{bmatrix} F_k^\mathrm{T} P_k F_k + \\ G_k^\mathrm{T} G_k - \gamma^2 I \end{bmatrix}^{-1} [A_k^\mathrm{T} P_k F_k + C_k^\mathrm{T} G_k]^\mathrm{T} \right\}$$

$$+K_k^{\mathrm T}\left\{B_k^{\mathrm T}P_kA_k+D_k^{\mathrm T}C_k-\begin{pmatrix}B_k^{\mathrm T}P_kF_k+\\ D_k^{\mathrm T}G_k\end{pmatrix}\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}\begin{pmatrix}A_k^{\mathrm T}P_kF_k+\\ C_k^{\mathrm T}G_k\end{pmatrix}^{\mathrm T}\right\}$$

$$+\left\{A_k^{\mathrm T}P_kB_k+C_k^{\mathrm T}D_k-(A_k^{\mathrm T}P_kF+C_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\}K_k$$

$$+K_k^{\mathrm T}\left\{R_k+B_k^{\mathrm T}P_kB_k+D_k^{\mathrm T}D_k-(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\}K_k. \qquad (17)$$

Equivalently, the equation can be simply written as

$$P_k=\Upsilon_k+K_k^{\mathrm T}\Omega_k^{\mathrm T}+\Omega_kK_k+K_k^{\mathrm T}\Phi_kK_k, \qquad (18)$$

where

$$\Upsilon_k=\left\{A_k^{\mathrm T}P_kA_k+C_k^{\mathrm T}C_k+Q_k-[A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k]\cdot\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}[A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k]^{\mathrm T}\right\},$$

$$\Omega_k=\left\{A_k^{\mathrm T}P_kB_k+C_kD_k-(A_k^{\mathrm T}P_kF+C_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\},$$

$$\Phi_k=\left\{R_k+B_k^{\mathrm T}P_kB_k+D_k^{\mathrm T}D_k-(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\}. \qquad (19)$$

By completing the square in the controller gain K_k, we have

$$P_k=\Upsilon_k+(K_k-K_k^o)^{\mathrm T}\Phi_k(K_k-K_k^o)-K_k^{o\mathrm T}\Phi_kK_k^o. \qquad (20)$$

For Equation (18) to be equal to Equation (20), we must have

$$-K_k^{o\mathrm T}\Phi_kK_k=\Omega_kK_k. \qquad (21)$$

Therefore, the optimal feedback gain

$$K_k^o=-\Phi_k^{-1}\Omega_k^{\mathrm T}$$
$$=-\left\{R_k+B_k^{\mathrm T}P_kB_k+D_k^{\mathrm T}D_k-\begin{pmatrix}B_k^{\mathrm T}P_kF_k\\ +D_k^{\mathrm T}G_k\end{pmatrix}\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}\begin{pmatrix}B_k^{\mathrm T}P_kF_k\\ +D_k^{\mathrm T}G_k\end{pmatrix}^{\mathrm T}\right\}^{-1}$$
$$\times\left\{B_k^{\mathrm T}P_kA_k+D_k^{\mathrm T}C_k-\begin{pmatrix}B_k^{\mathrm T}P_kF_k+\\ D_k^{\mathrm T}G_k\end{pmatrix}\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}\begin{pmatrix}A_k^{\mathrm T}P_kF_k+\\ C_k^{\mathrm T}G_k\end{pmatrix}^{\mathrm T}\right\}. \qquad (22)$$

When $K_k=K_k^o$, the minimum P_k is defined by the positive-definite solution of the following generalized SDRE:

$$P_k=\Upsilon_k-K_k^{o\mathrm T}\Phi_kK_k^o=$$
$$\left\{A_k^{\mathrm T}P_kA_k+C_k^{\mathrm T}C_k+Q_k-[A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k]\cdot\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}[A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k]^{\mathrm T}\right\}$$
$$-\left\{A_k^{\mathrm T}P_kB_k+C_k^{\mathrm T}D_k-(A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\}$$
$$\times\left\{R_k+B_k^{\mathrm T}P_kB_k+D_k^{\mathrm T}D_k-(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)^{\mathrm T}\right\}^{-1}$$
$$\times\left\{B_k^{\mathrm T}P_kA_k+D_k^{\mathrm T}C_k-(B_k^{\mathrm T}P_kF_k+D_k^{\mathrm T}G_k)\times\begin{bmatrix}F_k^{\mathrm T}P_kF_k+\\ G_k^{\mathrm T}G_k-\gamma^2I\end{bmatrix}^{-1}(A_k^{\mathrm T}P_kF_k+C_k^{\mathrm T}G_k)^{\mathrm T}\right\}. \qquad (23)$$

Equation (23) is the generalized discrete SDRE equation. By solving P_k from Equation (23), the H_2–H_∞ SDRE control can be achieved by Equation (22). ■

Remark 1: As a special case, if there is no H_∞ component in the performance index, i.e. the problem is of nonlinear quadratic regulator control, then the following controller can be derived as a special case of the above results:

By neglecting the noise term, the system equation becomes

$$x_{k+1} = A_k x_k + B_k u_k. \tag{24}$$

The optimal feedback control gain as

$$K_k^o = -\{R_k + B_k^T P_k B_k\}^{-1} B_k^T P_k A_k, \tag{25}$$

where P_k is defined by the positive-definite solution of the following generalized SDRE:

$$P_k = A_k^T P_k A_k - \{A_k^T P_k B_k\}\{R_k + B_k^T P_k B_k\}^{-1}\{B_k^T P_k A_k\} + Q_k. \tag{26}$$

Therefore, the conventional discrete SDRE solution (Dutka et al., 2005) is derived as a special case of our results.

Remark 2: The generalized SDRE (23) can be numerically difficult to solve. To facilitate the computation process, the following two results provide two alternative numerical solutions to the generalized SDRE in Theorem 1. Method 1 provides us the solution by solving the difference SDRE (28) until the steady state is reached, instead of (23). Method 2 provides us a state-dependent linear matrix inequality approach.

Numerical method 1 (H_2–H_∞ difference SDRE control)

Given the system (1), performance output (2), control input (3) and performance index (6), optimality can be achieved by using the control feedback

$$K_k = -\left\{R_k + B_k^T P_k B_k + D_k^T D_k - \begin{pmatrix} B_k^T P_k F_k \\ +D_k^T G_k \end{pmatrix} \times \begin{bmatrix} F_k^T P_k F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{pmatrix} B_k^T P_k F_k \\ +D_k^T G_k \end{pmatrix}^T \right\}^{-1} \times \left\{ B_k^T P_k A_k + D_k^T C_k - (B_k^T P_k F_k + D_k^T G_k) \times \begin{bmatrix} F_k^T P_k F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} (A_k^T P_k F_k + C_k^T G_k)^T \right\}, \tag{27}$$

where P_k is obtained as the steady solution to the following difference SDRE equation:

$$P_{k,i+1} = \left\{ A_k^T P_{k,i} A_k + C_k^T C_k + Q_k - \begin{bmatrix} A_k^T P_{k,i} F_k \\ +C_k^T G_k \end{bmatrix} \cdot \begin{bmatrix} F_k^T P_{k,i} F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} A_k^T P_{k,i} F_k \\ +C_k^T G_k \end{bmatrix}^T \right\} - \left\{ A_k^T P_{k,i} B_k + C_k^T D_k - (A_k^T P_{k,i} F_k + C_k^T G_k) \times \begin{bmatrix} F_k^T P_{k,i} F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} (B_k^T P_{k,i} F_k + D_k^T G_k)^T \right\} \times \left\{ R_k + B_k^T P_{k,i} B_k + D_k^T D_k - (B_k^T P_{k,i} F_k + D_k^T G_k) \times \begin{bmatrix} F_k^T P_{k,i} F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} (B_k^T P_{k,i} F_k + D_k^T G_k)^T \right\}^{-1} \times \left\{ B_k^T P_{k,i} A_k + D_k^T C_k - (B_k^T P_{k,i} F_k + D_k^T G_k) \times \begin{bmatrix} F_k^T P_{k,i} F_k + \\ G_k^T G_k - \gamma^2 I \end{bmatrix}^{-1} (A_k^T P_{k,i} F_k + C_k^T G_k)^T \right\}. \tag{28}$$

At time step k, the difference equation (28) is iterated starting with an arbitrary initial condition $P_{k,0} > 0$ until $P_{k,i}$ converges to $P_{k,i+1}$, for $i = 1, 2, 3, \ldots$. Hence, the solution to the generalized SDRE equation (23) can be found using this method. In practical applications, we can choose

$$P_{k,0} = I \tag{29}$$

as the starting value for iterations to calculate P_k.

Numerical method 2 (state-dependent LMI control)

Given the system equation (1), performance output (2), control input (3) and performance index (6), if there exist matrices $M_k = P_k^{-1} > 0$ and Y_k for all $k \geq 0$, such that the following state-dependent LMI holds (Wang, Yaz, & Long, 2014a, 2014b):

$$\begin{bmatrix} M_k & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_k & 0 & 0 & 0 \\ * & * & * & I_n & 0 & 0 \\ * & * & * & * & I_m & 0 \\ * & * & * & * & * & I_p \end{bmatrix} \geq 0, \tag{30}$$

where

$$\Xi_{12} = -\alpha M_k C_k^T D_k + 0.5 \cdot \beta M_k C_k^T,$$
$$\Xi_{13} = M_k A_k + Y_k^T B_k^T,$$
$$\Xi_{14} = M_k Q_k^{T/2},$$
$$\Xi_{15} = Y_k^T R_k^{T/2},$$

$$\Xi_{16} = \alpha^{1/2} M_k C_k^{\mathrm{T}},$$
$$\Xi_{22} = -\gamma I - \alpha D_k^{\mathrm{T}} D_k + 0.5 \cdot \beta (D_k + D_k^{\mathrm{T}}),$$
$$\Xi_{23} = F_k^{\mathrm{T}}, \tag{31}$$

$$\text{and} \quad M_{k+1} \geq M_k, \text{ where } \max \pi_k \text{ s.t. } M_k \geq \pi_k I, \tag{32}$$

then inequality (5) is satisfied. The nonlinear feedback gain of the controller is given by

$$K_k = Y_k \cdot M_k^{-1}. \tag{33}$$

Proof: Inequality (10) is equivalent to the $\Psi \leq 0$ following inequality:

$$\begin{bmatrix} \begin{pmatrix} P_k - Q_k - K_k^{\mathrm{T}} R_k K_k - \\ (C_k + D_k K_k)^{\mathrm{T}} (C_k + D_k K_k) \end{pmatrix} & -(C_k + D_k K_k)^{\mathrm{T}} G_k \\ * & \gamma^2 I - G_k^{\mathrm{T}} G_k \end{bmatrix} - \begin{bmatrix} (A_k + B_k K_k)^{\mathrm{T}} \\ F_k^{\mathrm{T}} \end{bmatrix} P_{k+1} [(A_k + B_k K_k) \quad F_k] \geq 0. \tag{34}$$

By adding and subtracting the same term in Equation (34), the following inequality results:

$$\begin{bmatrix} \begin{pmatrix} P_k - Q_k - K_k^{\mathrm{T}} R_k K_k - \\ (C_k + D_k K_k)^{\mathrm{T}} (C_k + D_k K_k) \end{pmatrix} & -(C_k + D_k K_k)^{\mathrm{T}} G_k \\ * & \gamma^2 I - G_k^{\mathrm{T}} G_k \end{bmatrix} - \begin{bmatrix} (A_k + B_k K_k)^{\mathrm{T}} \\ F_k^{\mathrm{T}} \end{bmatrix} (P_{k+1} - P_k) [A_k + B_k K_k \quad F_k] - \begin{bmatrix} (A_k + B_k K_k)^{\mathrm{T}} \\ F_k^{\mathrm{T}} \end{bmatrix} P_k [(A_k + B_k K_k) \quad F_k] \geq 0, \tag{35}$$

Therefore, subject to $P_{k+1} \leq P_k$, Equation (35) can be rewritten as

$$\begin{bmatrix} \begin{pmatrix} P_k - Q_k - K_k^{\mathrm{T}} R_k K_k - \\ (C_k + D_k K_k)^{\mathrm{T}} (C_k + D_k K_k) \end{pmatrix} & -(C_k + D_k K_k)^{\mathrm{T}} G_k \\ * & \gamma^2 I - G_k^{\mathrm{T}} G_k \end{bmatrix} - \begin{bmatrix} (A_k + B_k K_k)^{\mathrm{T}} \\ F_k^{\mathrm{T}} \end{bmatrix} P_k [(A_k + B_k K_k) \quad F_k] \geq 0. \tag{36}$$

By applying the Schur complement result, we obtain

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ * & \Gamma_{22} & \Gamma_{23} \\ * & * & \Gamma_{33} \end{bmatrix} \geq 0, \tag{37}$$

where

$$\Gamma_{11} = P_k - Q_k - K_k^{\mathrm{T}} R_k K_k - (C_k + D_k K_k)^{\mathrm{T}} (C_k + D_k K_k),$$
$$\Gamma_{12} = -(C_k + D_k K_k)^{\mathrm{T}} G_k,$$
$$\Gamma_{13} = (A_k + B_k K_k)^{\mathrm{T}},$$
$$\Gamma_{22} = \gamma^2 I - G_k^{\mathrm{T}} G_k,$$
$$\Gamma_{23} = F_k^{\mathrm{T}},$$
$$\Gamma_{33} = P_k^{-1}. \tag{38}$$

By pre-multiplying and post-multiplying the matrix with block diagonal matrix diag$\{M_k, I, I\}$, where $M_k = P_k^{-1}$, the following inequality as follows:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & \Theta_{33} \end{bmatrix} \geq 0, \tag{39}$$

where

$$\Theta_{11} = M_k - M_k (Q_k + K_k^{\mathrm{T}} R_k K_k - (C_k + D_k K_k)^{\mathrm{T}} (C_k + D_k K_k)) M_k,$$
$$\Theta_{12} = -M_k (C_k + D_k K_k)^{\mathrm{T}} G_k = -M_k C_k^{\mathrm{T}} G_k - Y_k^{\mathrm{T}} D_k^{\mathrm{T}} G_k,$$
$$\Theta_{13} = M_k (A_k + B_k K_k)^{\mathrm{T}} = M_k A_k^{\mathrm{T}} + Y_k^{\mathrm{T}} B_k^{\mathrm{T}},$$
$$\Theta_{22} = \gamma^2 I - G_k^{\mathrm{T}} G_k,$$
$$\Theta_{23} = F_k^{\mathrm{T}},$$
$$\Theta_{33} = M_k. \tag{40}$$

Finally, by applying the Schur complement again, the following LMI result is obtained:

$$\begin{bmatrix} M_k & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_k & 0 & 0 & 0 \\ * & * & * & I_n & 0 & 0 \\ * & * & * & * & I_m & 0 \\ * & * & * & * & * & I_p \end{bmatrix} \geq 0, \tag{41}$$

where

$$\Xi_{12} = -M_k C_k^{\mathrm{T}} G_k - Y_k^{\mathrm{T}} D_k^{\mathrm{T}} G_k,$$
$$\Xi_{13} = M_k A_k^{\mathrm{T}} + Y_k^{\mathrm{T}} B_k^{\mathrm{T}},$$
$$\Xi_{14} = M_k Q_k^{\mathrm{T}/2},$$
$$\Xi_{15} = Y_k^{\mathrm{T}} R_k^{\mathrm{T}/2}, \tag{42}$$
$$\Xi_{16} = M_k (C_k + D_k K_k)^{\mathrm{T}} = M_k C_k^{\mathrm{T}} + Y_k^{\mathrm{T}} D_k^{\mathrm{T}},$$
$$\Xi_{22} = \gamma^2 I - G_k^{\mathrm{T}} G_k,$$
$$\Xi_{23} = F_k^{\mathrm{T}},$$

Hence, if the LMI (41) holds, inequality (5) is satisfied. ■

Remark 3: Maximizing π_k in Equation (32) minimizes a bound on P_k and therefore forces the solution to be close to the one given in the SDRE in Theorem 1.

Remark 4: If the generalized SDRE (23) cannot be solved, then methods 1 and 2 provide alternative solutions to the generalized SDRE.

Remark 5: The solution of the SDLMI in method 2 involves successive LMI solutions and each solution depends on the measured state.

$H_2{-}H_\infty$ SDRE Control of inverted pendulum on a CART

The inverted pendulum on a cart problem (Wang & Yaz, 2009; Wang, Yaz, & Jeong, 2010; Wang, Yaz, & Jeong, 2010) is a classical control problem used widely as a benchmark for testing control algorithms. It is used here to demonstrate the effectiveness of the $H_2{-}H_\infty$ SDRE control approach. Traditional nonlinear control techniques assume that θ is a very small angle, $\cos(\theta) \cong 1$ and $\sin(\theta) \cong 0$, then $\cos(\theta) \cong 1$, $\sin(\theta) \cong 0$ linearize the system equation around its equilibrium point and apply the linear control techniques. However, it can be shown that the traditional control is not guaranteed to be optimal or stable. In this paper, we will not resort to the linearization approach. A model of the inverted pendulum problem can be derived using standard techniques:

$$\begin{aligned} (M+m)\ddot{x} + b\dot{x} + mL\ddot{\theta}\cos(\theta) - mL\dot{\theta}^2\sin(\theta) &= F, \\ (I+mL^2)\ddot{\theta} + mgL\sin(\theta) + mL\ddot{x}\cos(\theta) &= 0, \end{aligned} \tag{43}$$

where M is the mass of the cart, m the mass of the pendulum, b the friction coefficient between cart and ground, L the length to the pendulum centre of mass (length of the pendulum equals $2L$), $I = (1/3)m(2L)^2$ the inertia of the pendulum and F the external force, input to the system.

Denote the following state variables:

$$x_{1,k} = x(kT), \quad x_{2,k} = \dot{x}(kT), \quad x_{3,k} = \theta(kT) \text{ and } x_{4,k} = \dot{\theta}(kT).$$

By applying the Euler discretization method with sampling period T, and using the notation

$$\begin{aligned} \Omega_1 &= I + mL^2 - \frac{m^2L^2\cos^2(x_{3,k})}{M+m}, \\ \Omega_2 &= M + m - \frac{m^2L^2\cos^2(x_{3,k})}{I+mL^2} \end{aligned} \tag{44}$$

the discrete-time system equation can be written as

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & T \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} u_k, \tag{45}$$

where u_k is the kth sampling instant value of the input force F and

$$\begin{aligned} a_{22} &= 1 + T\frac{-b}{\Omega_2}, \\ a_{23} &= T\frac{m^2L^2 g\cos(x_{3,k})}{\Omega_2(I+mL^2)}\frac{\sin(x_{3,k})}{x_{3,k}}, \\ a_{24} &= T\frac{mL\sin(x_{3,k})}{\Omega_2}x_{4,k}, \\ a_{42} &= T\frac{mLb\cos(x_{3,k})}{(M+m)\Omega_1}, \\ a_{43} &= -T\frac{mgL}{\Omega_1}\frac{\sin(x_{3,k})}{x_{3,k}}, \\ a_{44} &= 1 - T\frac{m^2L^2\cos(x_{3,k})\sin(x_{3,k})x_{4,k}}{(M+m)\Omega_1}, \\ b_2 &= \frac{T}{\Omega_2}, \\ b_4 &= -T\frac{mL\cos(x_{3,k})}{(M+m)\Omega_1}. \end{aligned} \tag{46}$$

It should be noted that this state space formulation does not involve a process of linearization, but a process of state-dependent modelling. To avoid the division by zero, the term $\sin(x_{3,k})/x_{3,k}$ is substituted for $x_{3,k} = 0$ by the limit $\lim_{x_{3,k}\to 0}(\sin(x_{3,k})/x_{3,k}) = 1$.

The following system parameters are assumed:

$$M = 0.5\text{ kg}, \quad m = 0.5\text{ kg}, \quad b = 0.1\text{ N}\cdot\frac{\text{sec}}{\text{m}}, \quad L = 0.3\text{ m} \quad \text{and} \quad I = 0.06\text{ kg m}^2.$$

The following design parameters are chosen to satisfy different mixed criteria:

Classical SDRE Design (NLQR only)

$$C = [1 \quad 1 \quad 1 \quad 1], \quad D = [1], \quad Q = I_4 \quad \text{and} \quad R = 1.$$

$H_2{-}H_\infty$ Difference SDRE Method (Difference SDRE)

$$C = [0.01 \quad 0.01 \quad 0.01 \quad 0.01],\ D = [0.1],\ G = [0.01],\ Q = I_4,\quad R = 0.5,\ \gamma^2 = 0.01 \text{ and } P_0 = I_4.$$

State-dependent $H_2{-}H_\infty$ LMI Design (Predominant H_2)

$$C = [0.01 \quad 0.01 \quad 0.01 \quad 0.01],\ D = [0.01],\ G = [0.01],$$
$$Q = I_4, \quad R = 1 \quad \text{and} \quad \gamma^2 = 5.$$

State-dependent H_2–H_∞ LMI Design (Predominant H_∞)

$$C = [1 \quad 1 \quad 1 \quad 1],\ D = [1],\ G = [1],\ Q = 0.01 \times I_4,$$
$$R = 0.01 \quad \text{and} \quad \gamma^2 = 5.$$

LQR-based on linearization

$$Q = \text{diag}([10,\ 10,\ 50,\ 2]), \qquad R = 1.$$

The following initial conditions are assumed:

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = \pi/4 \quad \text{and} \quad x_4 = 0.$$

Simulation results for different design parameter values are compared in Figures 1–5 for performance: the classical SDRE or NLQR result (Dutka et al., 2005), the new H_2–H_∞ controller for a set of design parameter values computed by using the difference equation technique, new controller for two different sets of parameter

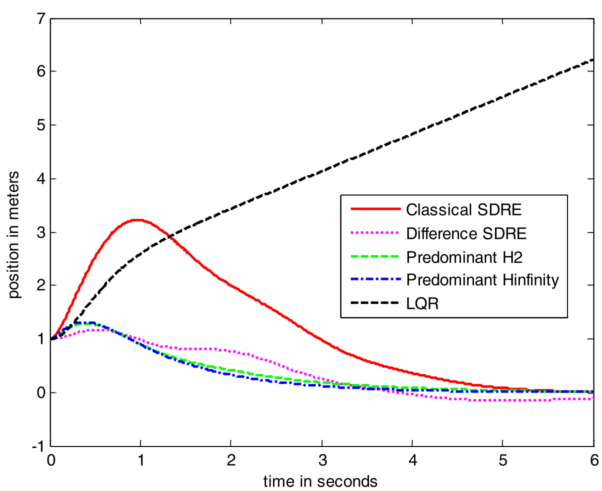

Figure 1. Position trajectory of the inverted pendulum.

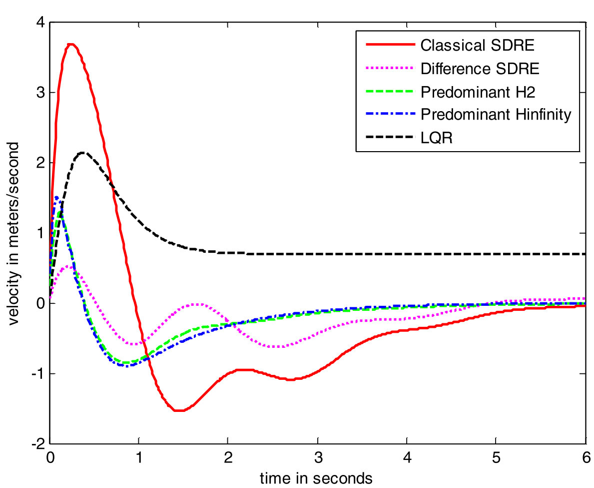

Figure 2. Velocity trajectory of the inverted pendulum.

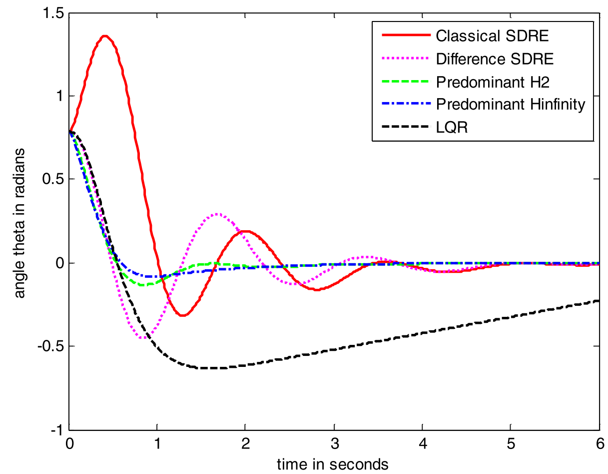

Figure 3. Angle ‘theta’ trajectory of the inverted pendulum.

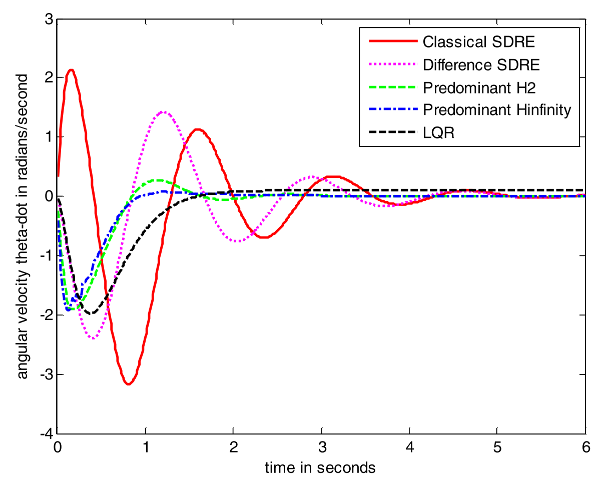

Figure 4. Angular velocity trajectory of the inverted pendulum.

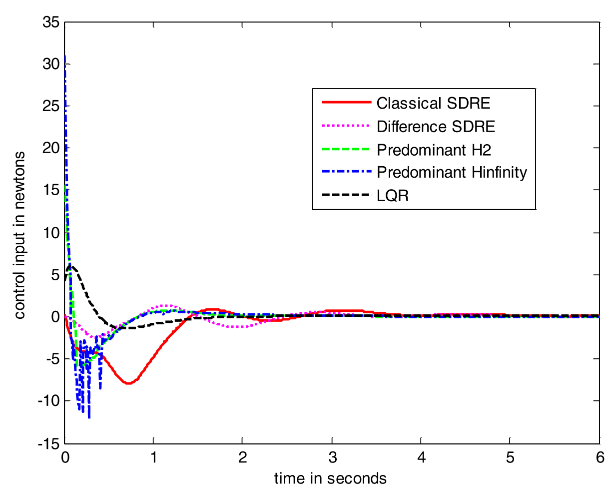

Figure 5. Control input.

values computed by the SDLMI technique and the traditional LQR control based on linearization. From these results, one can choose the controller that suits the designer's expectation best. Note that Figures 1, 3 and 4 show that the traditional LQR technique loses control of the state variables. Figure 5 shows that the lowest control magnitude is needed by the linearization-based LQR technique at the expense of losing control of the state trajectory.

Conclusions

A novel H_2-H_∞ control of discrete-time nonlinear systems with SDRE approach is presented in this paper. The optimal control solution can be obtained by solving generalized state-dependent Riccati equations or state-dependent LMIs. The inverted pendulum on a cart is used as an illustrative example. For future work, the mixed H_2-H_∞ SDRE control approach will be extended to nonlinear systems with nonaffine structure.

Disclosure statement

No potential conflict of interest was reported by the authors.

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