Enots Wolley Variations and related sequences

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ENOTS WOLLEY VARIATIONS AND RELATED SEQUENCES

by

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ABSTRACT
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The Enots Wolley sequence is a lexicographically earliest sequence (LES) that is closely related to the Yellowstone sequence. It is an open conjecture by N. J. Sloane that every number with at least two distinct prime factors appears as a term of the Enots Wolley sequence. In this thesis, this conjecture is proved for a variation of the Enots Wolley sequence that operates on the binary representation of a positive integer rather than the prime factorization. The methods used are then applied to prove some new properties of the prime factorization Enots Wolley sequence.

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CHAPTER 1
INTRODUCTION

1.1 Lexicographically earliest sequences

There are many integer sequences naturally arising in mathematics and on the Online Encyclopedia of Integer Sequences (OEIS) that are defined as *lexicographically earliest sequences* (LESs) [6]. The concept of a LES has no rigorous definition for reasons that will be made clear, but instead informally refers to an approach to constructing an integer sequence that possesses some set of prescribed properties. Often, LESs are interesting in their own right as they may possess non-trivial emergent properties.

The Stanley sequence $S(1)$ (OEIS number A005836) is one of the most well-known and earliest examples of a sequence of this type [5]. Its definition is as follows:

Definition 1. For $k \in \mathbb{Z}_{>0}$, a *k-arithmetic progression* (k-AP) is a set of k natural numbers of the form $\{c + di \mid 0 \leq i \leq k - 1\}$ for some constants $c, d \in \mathbb{Z}_{\geq 0}$.

Definition 2 (A. M. Odlyzko and R. P. Stanley [5])**.** $S(1)$ is defined as the lexicographically earliest increasing sequence starting $0, 1, \ldots$ that contains no 3-arithmetic progressions.

For context, Roth's theorem on arithmetic progressions states that any that set of natural numbers with positive upper density contains a 3-AP. [1] Stanley and Odlyzko studied the sequence $S(1)$ as they were interested in the least possible rate of growth of a sequence which avoids 3-APs. Although it turns out that $S(1)$ grows faster than other known sequences avoiding 3-APs, they realized that this sequence has its own fascinating properties. Namely, the following theorem:

[1] Readers interested in a more complete treatment of this subject, see [2]

Theorem 3 (A. M. Odlyzko and R. P. Stanley [5])**.** *A number* $k \in \mathbb{Z}_{\geq 0}$ *appears as a term of the sequence* $S(1)$ *if and only if the base* 3 *representation of* k *does not contain the digit* 2*.*

Sequences such as the Stanley sequence $S(1)$ that are defined as the LES satisfying some easily-stated property P belong to the informal class of sequences often referred to as *LESs*. LESs should be understood as a way of thinking about all integer sequences rather than a distinct type of integer sequence. Indeed, it will soon be shown that all sequences of positive integers qualify as LESs under the natural definition the term. Despite being a vague concept, there is a useful shared terminology surrounding LESs as well as a few basic results that transfer between specific types of LESs.

To understand why it is better to avoid rigorously defining the concept of a LES, consider the following naïve definition:

Definition 4. A sequence of positive integers $(a_i)_{i\geq 0}$ is a *LES* if it is the lexicographically earliest sequence satisfying some property P.

As the following theorem demonstrates, this definition is naïve because it includes every sequences of positive integers:

Theorem 5. *Let* $(a_i)_{i\geq 0}$ *be any sequence of positive integers. Then,* a_i *is a LES.*

Proof. It suffices to produce a property P_0 of integer sequences such that a_i is the lexicographically earliest sequence satisfying property P_0. For this, say that an integer sequence $(c_i)_{i\geq 0}$ has property P_0 if $c_i = a_i$ for all integers i. Then, a_i is trivially the lexicographically earliest sequence satisfying property P_0 as it is the only sequence satisfying property P_0. □

Thus, any integer sequence can be studied in terms of being the LES satisfying some property P. In practice, however, the term LES is usually used to refer to sequences whose "most natural" definition is as a LES among some infinite set of sequences. Often, these types of sequence have no known closed form or alternative definition, although this is not a requirement.

In the proof of Theorem 5, the property P_0 is maximally restrictive in the sense that it is only satisfied by a single sequence. Naturally, more interesting LESs arise from creative choices of the property P. Considering the LES satisfying P is a technique to isolate a single extremal sequence out of a large class of sequences.

Another reason that Definition 4 is naïve is because of complications arising from the fact that the lexicographical order is not a well-ordering of the set of infinite sequences of positive integers. In particular, even in cases where there are sequences that satisfy the defining property P, it may be the case that there is no LES satisfied by P. This is best illustrated via the following example.

Example 6. Say that a sequence of positive integers $(a_i)_{i\geq 1}$ satisfies P if $(a_i)_{i\geq 1}$ consists of a block of finitely many 1s following by infinity 2s. Then, set of sequences satisfying P, in descending lexicographical order, begins

$$\begin{gathered} 2,2,2,2,\ldots \\ 1,2,2,2,\ldots \\ 1,1,2,2,\ldots \\ 1,1,1,2,\ldots \\ \vdots \end{gathered} \tag{1.1}$$

Observe that there is no LES satisfying the property P.

The remainder of this section is devoted to introducing concepts shared among all sequences studied as LESs. In particular, there is a close relationship between LESs and greedy algorithms.

Definition 7. Suppose $(a_i)_{i\geq 0}$ is the LES satisfying property P. Let $k \in \mathbb{Z}_{\geq 0}$ be a natural number. If there is a sequence $(c_i)_{i\geq 0}$ satisfying property P such that

$$\begin{aligned} c_i &= a_i \text{ for all } i < n \\ c_n &= k \end{aligned} \tag{1.2}$$

say that k is a *candidate* for the nth term of a_i.

If $(a_i)_{i\geq 0}$ is the LES satisfying property P, an algorithm to decide whether or not k is a candidate for the nth term yields an algorithm to compute arbitrary terms of the sequence.

Algorithm 1 Compute the nth term of a LES

```
  Input n ≥ 0
  Output a_n, the nth term of the sequence
i ← 0
while True do
    if i is a candidate for a(n) then
        return i
    end if
    i ← i + 1
end while
```

Theorem 8. *If $(a_i)_{i\geq 1}$ is the LES $(a_i)_{i\geq 1}$ satisfying property P, then Algorithm 1 computes the terms of $(a_i)_{i\geq 1}$.*

Proof. By Definition 7, $a(n)$ itself is always the least candidate for the nth term of a_i. Because there are no candidates for the nth term smaller than $a(n)$, the while loop of Algorithm 1 executes exactly $a(n)$ times and then returns $a(n)$. □

Because Algorithm 1 computes the nth term by returning the least candidate, it can be thought of as a greedy algorithm (despite the concept of a "greedy algorithm" being ill-defined.) For this reason, LESs are sometimes referred to as "greedy sequences" or arise in the study of some greedy algorithm.

To use Algorithm 1, one must only implement a function $\mathrm{CAND}((a_1, \ldots, a_{n-1}), i)$ to decide whether $i \in \mathbb{Z}_{\geq 0}$ is a candidate for the nth term given a partial sequence $a_1, \ldots, a_{n-1}$. Depending on the sequence, the resulting algorithm may be slow or it might be possible to devise a faster algorithm. Still, this is often enough to compute the first few terms of the sequence with minimal effort. In practice, it is often clear that the LES satisfying P exists.

As previously stated, LESs should be thought of as a set of shared concepts or techniques rather than a specific subset of integer sequences. An important implication of Algorithm 1 is a technique for experimentally discovering new LESs. Namely, it is often easy to come up with choices of CAND that produce interesting sequences with complex emergent behavior. For many sequences discovered in this way, Algorithm 1 is the only known method for computing the terms.

For particular classes of LESs, it may be possible to make improvements to Algorithm 1. Of particular interest is the case of injective LESs, which will be discussed in Section 1.3.

1.2 The EKG family of injective LESs

To summarize, the high-level idea of a LES is to start with some defining property P and then study the LES that satisfies P. For the types of LESs that will be discussed in this thesis, it will be the case that the defining property P is "local" in the sense that it is a property that must hold for all length k blocks of consecutive terms. For such sequences, it is natural to study non-trivial global properties that emerge. Often, the foremost such global property is the set of numbers appearing in the sequence. Theorem 3 is an example of this type of result, although the definition of $S(1)$ is not "local" in this sense. An important example of a LES of this type is the EKG sequence, which is defined as follows.

Definition 9. A sequence $(a(n))_{i\geq 1}$ of positive integers is *injective* if and only if it does not have any repeated terms.

Definition 10 (A064413, Jonathan Ayres)**.** The *EKG sequence* $(a(n))_{i\geq 1}$ is defined as the lexicographically earliest injective sequence starting with $a(1) = 1, a(2) = 2$ such that for all $n > 2$, $\gcd(a(n-1), a(n)) \neq 1$.

The EKG sequence is primarily interesting because it exhibits surprising emergent phenomena. Namely, the plot exhibits consistent spikes reminiscent of an electrocardiogram. Proving this easily observable property requires some ingenuity. Lagarias, Rains and Sloane [4] proved that every positive integer appears in the sequence. Further results concerning the EKG sequence such as upper and lower bounds were later proved by Hofman and Pilipczuk [3].

The EKG sequence is also significant because it inspired a family injective LESs including the Yellowstone sequence (A098550,) the Enots Wolley sequence (A336957) and the as-yet unnamed sequence A350359 among others. What these injective LESs have in common is that their defining properties involve restrictions on the common divisors of consecutive terms. Definitions of these sequences will be provided, but first it is necessary to introduce a concept shared among this family of sequences.

Definition 11. Let k be a positive integer. Let $\operatorname{supp}(k)$, the *support* of k, denote the set of primes in the prime factorization of k with multiplicity discarded.

For example, $\text{supp}(15) = \{3, 5\}$, $\text{supp}(1) = \emptyset$ and $\text{supp}(12) = \{2, 3\}$. It is now possible to define the three sequences previously mentioned.

Definition 12 (Yellowstone permutation, A098550, Reinhard Zumkeller)**.** The *Yellowstone sequence* is defined as the injective LES starting with $a(1) = 1, a(2) = 2$ such that for all $n > 2$, $\text{supp}(a(n)) \cap \text{supp}(a(n-1)) = \emptyset$ and $\text{supp}(a(n)) \cap \text{supp}(a(n-2)) \neq \emptyset$.

For reference, the first few terms of the Yellowstone sequence are $1, 2, 3, 4, 9, 8, 15, 14, 5, 6$. The Yellowstone sequence can be described as the EKG sequence with the additional requirement that $\text{supp}(a(n))$ must be disjoint with $\text{supp}(a(n-2))$. One notable aspect of the Yellowstone sequence is that results similar to Theorem 3 exist for the Yellowstone sequence. In particular, it has been proved that all positive integers appear as a term [1]. The name "Yellowstone" comes from the observation that the plot of the sequence has predictable peaks and valleys at regular intervals, which are reminiscent of the periodic eruptions of the Old Faithful geyser in Yellowstone National Park.

By reversing the order of the two defining constraints of the Yellowstone sequence, one obtains the *Enots Wolley* sequence ("Enots Wolley" is "Yellow Stone" spelled backward.)

Definition 13 (Enots Wolley sequence, A336957, Scott R. Shannon and N. J. Sloane)**.** The *Enots Wolley* sequence is defined as the injective LES starting with $a(1) = 1, a(2) = 2$ such that for all $n > 2$, $\text{supp}(a(n)) \cap \text{supp}(a(n-2)) = \emptyset$ and $\text{supp}(a(n)) \cap \text{supp}(a(n-1)) \neq \emptyset$.

An implication of this definition is that only positive integers k with $|\text{supp}(k)| \geq 2$ can appear as terms of the Enots Wolley sequence. Whether or not all positive integers k with $\text{supp}(k) \geq 2$ actually do appear as terms of the Enots Wolley sequence is still the subject of an open conjecture.

Conjecture 14 (N. J. Sloane)**.** Every positive integer that is not a prime power appears as a term of the Enots Wolley sequence.

The Yellowstone and Enots Wolley sequences may be considered direct descendants of the EKG sequence. The EKG sequence has many more distant relatives. Here is just one of many examples:

Definition 15 (A350359, David James Sycamore)**.** The injective LES starting with $a(1) = 1, a(2) = 2$ such that for any four consecutive terms t_1, t_2, t_3, t_4, $\text{supp}(t_4)$ is disjoint with $\text{supp}(t_1)$ and $\text{supp}(t_3)$ but not $\text{supp}(t_2)$.

From here on, the phrase *number-theoretic EKG family* will be used to refer to any injective LESs defined by some divisibility constraint on blocks of consecutive terms. The meaning of "divisibility constraint" is intentionally vague as to not rule out LESs whose definitions involve different types of constraints on the supports of blocks of terms. It is not apparent how a formal definition would be useful as it is not clear that the number-theoretic EKG family has any common properties or results that transfer between distinct members.

A notable characteristic of the number-theoretic EKG family is that each member has several variations obtained by modifying the definition of the function supp (as defined in Definition 11.) For example, the function "supp" could alternatively be defined as follows:

Definition 16. Let k be a positive integer. Let $\text{supp}_2(k)$, the *binary support* of k, denote the set of positions of 1s in the binary expansion of k.

For example, because $13 = 2^3 + 2^2 + 2^0$ is the binary expansion of 13, $\text{supp}_2(13) = \{0, 2, 3\}$. Because $1 = 2^0$ is the binary expansion of 1, $\text{supp}_2(1) = \{0\}$.

By replacing "supp" with "supp_2" in Definition 12, Definition 13 or Definition 15 (plus adjusting the initial conditions to make this work), one obtains a derived sequence called the *binary analog*. The set of all binary analogs forms a family of sequences that will be referred to as the *binary EKG family*. As a motivating example, here is the definition of the binary analog of the Enots Wolley sequence:

Definition 17 (Binary Enots Wolley sequence, A338833, Nathan Nichols)**.** The *binary Enots Wolley Sequence* is defined as the injective LES starting with $a(1) = 1, a(2) = 3$ such that for all $n > 2$, $\mathrm{supp}_2(a(n)) \cap \mathrm{supp}_2(a(n-2)) = \emptyset$ and $\mathrm{supp}_2(a(n)) \cap \mathrm{supp}_2(a(n-1)) \neq \emptyset$.

This binary analog of the Enots Wolley sequence is the primary subject of this thesis. A more complete explanation of this sequence is included in Chapter 2, as well as several proofs. Here, the goal is to provide a brief overview of its significance.

The binary Enots Wolley sequence was defined by this author after several attempts at proving Conjecture 14 failed to produce results. The primary motivation for studying the binary Enots Wolley sequence was the hope that doing so would indirectly lead to progress towards Conjecture 14. The binary analog of Conjecture 14 turned out to be less difficult, and efforts to prove it were successful. These proofs are included in Chapter 2. In Chapter 3, new results for the original Enots Wolley sequence are derived by adapting the techniques used for the binary case. Thus, although Conjecture 14 remains open, studying the binary analog turned out to be a useful method for understanding the original.

This general strategy of studying the binary analog to understand the number-theoretic version is likely applicable to many other sequences of the number-theoretic EKG family. Another example of this technique is recent progress

	2	3	5	7	11	13	17	19	23	=
1										1
2	1									2
3	1	1								6
4		1	1							15
5			1	1						35
6	1			1						14
7	2	1								12
8		1			1					33
9			1		1					55
10	1		1							10
11	1	2								18
12		1		1						21
13				1	1					77
14	1				1					22
15	2		1							20
16		2	1							45
17		1				1				39
18	1					1				26
19	2			1						28
20		2		1						63
21		1					1			51
22	1						1			34
23	1							1		38
24		1						1		57
25		1							1	69
26	1								1	46
27	3		1							40
28			1			1				65
29				1		1				91
30	1	1		1						42
31	1	1	1							30
32			1				1			85
33				1			1			119
34	3			1						56

Enots Wolley

	1	2	4	8	16	32	64	=
1	1							1
2	1	1						3
3		1	1					6
4			1	1				12
5	1			1				9
6	1				1			17
7		1			1			18
8		1		1				10
9	1		1	1				13
10			1		1			20
11					1	1		48
12	1					1		33
13	1		1					5
14		1	1	1				14
15				1	1			24
16	1				1	1		49
17	1	1	1					7
18		1					1	66
19				1			1	72
20	1			1	1			25
21	1	1			1			19
22		1				1		34
23			1			1		36
24	1		1		1			21
25	1	1		1				11
26				1		1		40
27			1		1	1		52
28		1	1		1			22
29	1	1					1	67
30	1			1		1		41
31			1	1	1			28
32			1				1	68
33	1						1	65
34	1	1		1	1			27

Binary Enots Wolley

Figure 1.1: The prime factorizations of the first 34 terms of the Enots Wolley sequence and the binary expansions of the first 34 terms of the binary Enots Wolley sequence. A blank square indicates a multiplicity of zero or a coefficient of zero (respectively)

made on the binary analog of the Two-Up sequence [7]. Although conjectures concerning the number-theoretic Two-Up sequence (A090252) have long remained open, their counterparts for the binary analog (A354169) were proved along with other deep properties.

Although the number-theoretic versions of EKG-like sequences have traditionally received the most attention, the binary analogs are arguably just as interesting. In many cases, it is perhaps preferable to study the binary analog before trying to tackle the number-theoretic version because the binary analog is often more tractable.

In addition to the binary and number-theoretic EKG families, there are also entire families of EKG-like LESs that are as yet virtually unknown. Recall that the only difference between the number-theoretic and binary EKG families is the definition of the function $\text{supp}(k)$ that is used. There are many other possible ways to define the function $\text{supp}(k)$, and each possibility induces analogs of sequences such as the EKG sequence, the Yellowstone sequence, the Enots Wolley sequence and the Two-Up sequence among many others.

Intuitively, the function $\text{supp}(k)$ can be thought of as a generalized formulation of the concept of digits in some numeral system. The term *EKG-like sequence* will be used to refer to any injective LES whose defining condition operates only on the digits of a number with respect to some numeral system, and the phrase *analog EKG family* will be used to mean the set of variations of EKG-like sequences induced by a particular choice of numeral system. The number-theoretic EKG family and binary EKG family are examples of analog EKG families introduced previously.

For example, here is one possible definition of a support function corresponding to the usual base-10 numeral system.

Definition 18. Let k be a positive integer. Let $\text{supp}_{10}(k)$, the *decimal support* of k, denote the set of terms in the decimal expansion of k.

For example, because $246 = 200 + 40 + 6$, $\text{supp}_{10}(246) = \{200, 40, 6\}$.

Using this definition, it would be straightforward to define a decimal analog of any EKG-like sequence. The prime factorization of a natural number can also be thought of as a sort of number system that corresponds to the following support function.

Definition 19. Let k be a positive integer. Let $\text{supp}_p(k)$, the *full number-theoretic support* of k, denote the set of prime powers in the prime factorization of k.

For example, because $12 = 3 * 2^2$, $\text{supp}_p(12) = \{3, 2^2\}$.

These two examples of alternative support functions are relatively tame. There are a myriad of more exotic support functions that one could consider, and each defines an entire analog EKG family. From this perspective, the number-theoretic EKG family is only the tip of a great iceberg and it makes sense to study the lesser-known variations.

The advantage of thinking about numeral systems in terms of support functions is that the support function is independent of the actual meaning of the digits. To define an analog of some EKG-like sequence, it does not matter (for example) that the binary expansion is additive while the prime factorization is multiplicative. The only thing that the defining property of the sequence "sees" are the outputs of the support function. This setup is capable of handling more than just "numeral systems" in the traditional sense. For example, there is no requirement for the digit expansion to be unique (i.e., $\text{supp}(k)$ is not required to be injective.) Such is the case for $\text{supp}(k)$ as defined in Definition 11 since multiplicity is discarded.

A reoccurring theme among all analog EKG families is that it makes sense to think about such sequences in the format of Figure 1.1. In this figure as well as elsewhere in this thesis, the convention of placing the least significant digit first is adopted.

It is now necessary to state a few definitions that will be useful throughout the rest of this thesis. Here, $a(i)$ is some EKG-like sequence and the definition of the function $\operatorname{supp}(k)$ should be understood to be the same as what is used in the definition of $a(i)$.

Definition 20. For a positive integer k, the *characteristic function* $\operatorname{char}_k(i) : \mathbb{N} \to \{0, 1\}$ is

$$\operatorname{char}_k(i) := \begin{cases} 1 & \operatorname{supp}(k) \cap \operatorname{supp}(a(i)) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Definition 21. The *truth table* of $\operatorname{char}_k(i)$ is the sequence $(\operatorname{char}_k(i))_{i>0}$.

Definition 21 can be intuitively understood in terms of Figure 1.1. In the binary case, the truth table of $\operatorname{char}_{2^k}(i)$ is the sequence of values in the kth column of the table read from top to bottom.

Definition 22. For a non-negative integer k, let $\omega(k) := |\operatorname{supp}(k)|$ denote the *weight* of k.

For the function $\operatorname{supp}_2(k)$ of Definition 16, $\omega(k)$ is the Hamming weight (A000120.) For the function $\operatorname{supp}(k)$ of Definition 11, the function $\omega(k)$ is A001221.

1.3 Results on injective LESs

As previously discussed, there is not much general theory of LESs, injective LESs, or the EKG family. However, there are a couple minor results about injective

LESs that are worth mentioning for the purposes of this thesis. In particular, there is one important theorem that applies to all injective LESs, including all members of any analog EKG family. This theorem will be the basis for many of the proofs that will appear in later chapters.

Theorem 23. *Suppose* $(a_i)_{0\leq i}$ *is the LES satisfying property* P *and that* a_i *is injective. Suppose that* $S_k \subseteq \mathbb{Z}_{\geq 0}$ *is a set of natural numbers such that for all* $i \in S_k$*,* k *is a candidate for* $a(i)$*. Then, if* $|S_k| \geq k$*,* k *must appear in the sequence.*

Proof. If k is a candidate for the ith term but $a(i) \neq k$, it must be that $a(i) = k'$ where k' is another candidate for the ith term that is less than k. Because there are only $k-1$ positive integers k' less than k and each can only appear in the sequence once, this situation can only occur for at most $k-1$ choices of $i \in S_k$ before k must be the least candidate that has not already occurred in the sequence. □

As previously stated, it is often the case that we wish to understand the set of numbers appearing in a LES. Theorem 23 is thus significant as it provides a method for proving that a given number appears in an injective LES. Namely, it suffices to prove that k is a candidate for at least k distinct terms.

For LESs that arise in the course of research, typically the only known algorithm to compute terms is Algorithm 1. It is thus useful to note that Algorithm 1 can be improved in the special case of an injective LES. Namely, when computing the nth term of the sequence, it is justified to skip checking whether i is a candidate for the nth term if $a(m) = i$ for some $m < n$.

Algorithm 2 Compute the nth term of an injective LES

```
    Input n ≥ 0
    Output a_n, the nth term of the sequence
  i ← 0
  for Each i ≥ 0 not appearing in the sequence before the ith term do
      if i is a candidate for a(n) then
          return i
      end if
      i ← i + 1
  end for
```

The only aspect of Algorithm 2 that is dependent on the defining property P is the function to check whether i is a candidate for the nth term given a partial sequence $a_1, a_2, \ldots, a_{n-1}$. For any specific LES, it may be possible to use specific properties to improve this algorithm in other ways. Still, this simple algorithm is often sufficient for research purposes.

CHAPTER 2
THE BINARY ENOTS WOLLEY SEQUENCE

2.1 Basic properties

This section is concerned with the binary analog of the Enots Wolley sequence. The goal is to prove that every number with a binary weight of at least two appears in the sequence, which is the binary analog of Conjecture 14. Throughout this chapter, $a(n)$ shall always denote the binary Enots Wolley sequence of Definition 17 (A338833.)

The following theorem serves as a more verbose form of Definition 17. This is included primarily because it will be useful to refer to the properties (i), (ii) and (iii) directly. The proof also serves to establish a higher degree of familiarity with the definition of the binary Enots Wolley sequence.

Theorem 24. *For $n > 2$, $a(n)$ is the smallest number m not yet in the sequence such that:*

i) $\operatorname{supp}_2(m) \cap \operatorname{supp}_2(a(n-1))$ *is nonempty,*

ii) $\operatorname{supp}_2(m) \cap \operatorname{supp}_2(a(n-2))$ *is empty, and*

iii) $\operatorname{supp}_2(m) \not\subseteq \operatorname{supp}_2(a(n-1))$.

Proof. The binary Enots Wolley sequence is defined in Definition 17 as the LES satisfying some property P. It is immediate that the binary Enots Wolley sequence exists because 1 produces a partial sequence $(a_i)_{1 \leq i \leq n}$ that satisfies P for all n. This proves that the binary Enots Wolley sequence is well-defined.

What remains to be shown is that the sequence $a(n)$ satisfies properties (i), (ii) and (iii). Conditions (i) and (ii) are already part of Definition 17. Condition (iii) is a consequence of (i) and (ii) because if (iii) fails for some term $a(n)$, there is no way for property (ii) to hold for the term $a(n+1)$. This proves all candidates for any term $a(n)$ must satisfy (i)-(iii).

Finally, it is necessary to prove that all numbers satisfying properties (i)-(iii) for the term $a(n)$ are candidates. In other words, it must be shown there is no need for any additional rules to guarantee that Algorithm 2 does not lead to any dead ends. For this, observe that there will always be some way to continue the sequence given that the previous two terms satisfy (i)-(iii). In particular, property (iii) guarantees that it will always be possible to choose the next term. □

One of the implications of Theorem 24 is that only numbers with a binary weight of at least 2 can appear in the sequence. As the following theorem demonstrates, this lower bound is met when a digit appears in a term of the sequence for the first time:

Theorem 25. *If n is the least positive integer such that $v \in \operatorname{supp}_2(a(n))$, the term $a(n)$ has a binary weight of 2.*

Proof. Suppose n is the least positive integer such that $v \in \operatorname{supp}_2(a(n))$ and let S be the set of all candidates for the nth term. By Theorem 24, the term $a(n)$ is the least element of S which does not appear in the first $n-1$ terms of the sequence.

Define T to be the set of elements of $\operatorname{supp}_2(a(n-1))$ that are not also contained in $\operatorname{supp}_2(a(n-2))$. By Theorem 24, for every $m \in S$ there is at least one element of T contained in $\operatorname{supp}_2(m)$. The least element of S involving 2^v is $2^v + 2^w$ where w is the least element of T. Because the power of two 2^v is not used in the first $n-1$ terms of the sequence, the number $2^v + 2^w$ does not appear in the sequence prior to the nth term. Thus, $a(n) = 2^v + 2^w$. □

Theorem 25 justifies the following definition.

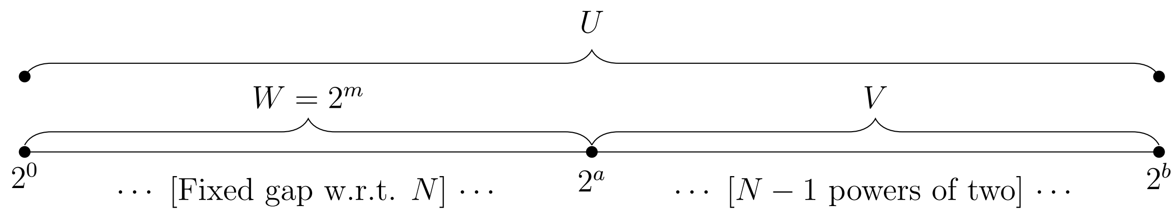

Figure 2.1: Visualization of the variables used in the proof of Theorem 27. Curly braces indicate the possible range of the corresponding power of two.

Definition 26. If $a(n) = 2^v + 2^w$ is the first term of the sequence such that $v \in \text{supp}_2(a(n))$, say that 2^v is *introduced* by 2^w.

2.2 Surjectivity of the Binary Enots Wolley sequence

The goal of this section is to prove that all positive integers m with Hamming weight at least 2 appear as a term in the binary Enots Wolley sequence. It is not feasible to directly find i such that $a(i) = m$ because there is a complicated dependence of the sequence on previous terms. Instead, the overall strategy will be to show that m must be a candidate for an infinite number of terms, thus appears in the sequence eventually by Theorem 23.

Theorem 27. *For all non-negative integers m, the truth table of* char_{2^m} *(i.e., the mth column of Figure 1.1) has infinitely many occurrences of* 01*.*

Proof. Since the truth table of a power of two cannot have any occurrences of 111 due to property (ii), it must only be shown that there is no i' such that $i > i'$ implies $\text{char}_{2^m}(i) = 0$. Assume that such an i' does exist. The goal is to prove that this assumption leads to a contradiction; namely, that there are further occurrences of 1 in the truth table of 2^m after the i'-th term.

Let 2^a be the largest power of two that has occurred in the sequence before the (i')th term. Let 2^b be a sufficiently large power of two so that $b - a > N$ where $N = N(2^m)$ is some number depending on 2^m. It will be shown that for any 2^m, it is possible to choose N sufficiently large so that 2^m occurs in the sequence at some point after the i'th term, contradicting the assumption that $\text{char}_{2^m}(i) = 0$ for all $i > i'$.

Let n be the least positive integer such that $b \in \text{supp}_2(a(n))$. It is now necessary to make the following three definitions:

- $U :=$ The power of two that introduces 2^b in the nth term.
- $V :=$ A power of two distinct from U such that $2^a < V < 2^b$.
- $W := 2^m$.

It will be shown that unless all of the $N - 1$ choices of V are present in $a(n-2)$, the number $U + V + W$ is a smaller unused candidate for $a(n)$ than $U + 2^b$.

First, consider the case where there is some possible choice of V that does not violate property (ii). It must be shown that $U + V + W < 2^b + U$. By subtracting U from both sides, this becomes $V + W < 2^b$. To prove this this inequality, observe that since V and W are distinct powers of two less than 2^b, there is no "carry" in the binary representation of the sum $V + W$.

Now, it has been established that it is possible to choose N such that $U + V + W < 2^b + U$. It remains to be shown that $U + V + W$ does not appear in the sequence before the nth term, and that $U + V + W$ satisfies the properties of Theorem 24 to be a candidate for the term $a(n)$. These requirements follow from the following properties of the three powers of two U, V and W:

- U: Ensures property (i) for $U + V + W$ to be a candidate for the nth term holds.
- V: Because W has never occurred together with V, this ensures $U + V + W$ has not appeared in the first $n - 1$ terms of the sequence.
- W: Ensures that if $a(n) = U + V + W$, then $\text{char}_{2^m}(n) = 1$ (contradicting the assumption that $\text{char}_{2^m}(n) = 0$,) and that property (iii) holds for $U + V + W$ to be a candidate for $a(n)$.

This proves that as long as V isn't barred from appearing in the term $a(n)$ by property (ii), the number $U+V+W$ is a smaller candidate for $a(n)$ that does not appear in the first $n-1$ terms of the sequence. In the case that all possible choices of V are present in $a(n-2)$, it is now claimed that it is possible to construct an alternative value of $a(n-2)$ that produces a lexicographically earlier sequence (which is a contradiction.) This construction is as follows:

Because $b-a>N$, there are $N-1$ possible choices for V. Hence, if all possible choices of V are present in $a(n-2)$, then $a(n-2)$ must have a Hamming weight of at least $N-1$. Since none of the powers of two that are possible choices for V have yet appeared in the sequence in a term together with 2^m, it is possible to erase one of the powers of two that are possible choices for V and replace it with 2^m in $a(n-2)$ to obtain a smaller unused candidate for the $(n-2)$nd term. □

Theorem 28. *Let k be a positive integer with $\omega(k) \geq 2$. If there are an infinite number of occurrences of* 01 *in the truth table of* $\text{char}_k(i)$*, then k must appear in the sequence.*

Proof. Let S be an infinite set of positive integers such that $\text{char}_k(j)=0$ and $\text{char}_k(j+1)=1$ for all $j \in S$. For each $j \in S$, properties (i) and (ii) of Theorem 24 always hold for k to be a candidate for the $(j+2)$nd term. If property (iii) for k to be a candidate for the $(j+2)$nd term also happens to hold for an infinite number of positions $j \in S$, then k is a candidate an infinite number of times and therefore must appear in the sequence by Theorem 23. In this case, there is nothing more to prove.

If property (iii) for k to be a candidate for the $(j+2)$nd term holds only for a finite number of $j \in S$, it must be that $\text{supp}_2(k)$ is a subset of $\text{supp}_2(a(j+1))$ an infinite number of times. Thus, all but finitely many of the $j \in S$ correspond to instances of the following pattern, where each of the m_0, m_1, m_2, m_3 have supports disjoint with $\text{supp}_2(k)$ and d is some positive integer such that $\text{supp}_2(d)$ is a (possibly empty) subset of $\text{supp}_2(k)$:

$$\begin{array}{ccccccccccc} & & (A) & & (B) & & (C) & & (D) & & \\ \cdots & \to & m_0 & \to & m_1 + k & \to & m_2 + d & \to & m_3 & \to & \cdots \end{array} \tag{2.1}$$

This means that almost every time there is a 1 after a run of 0s in the truth table of $\text{char}_k(i)$, all digits of k must be present in the term corresponding to the leading 1 (i.e., in position (B).) Note that the term in position (D) must have support disjoint with $\text{supp}_2(k)$. This implies that there can be almost no occurrences of 111 in truth table of $\text{char}_k(i)$.

Let p, q be two distinct elements of $\text{supp}_2(k)$. Let P_i and Q_i denote the set of distinct positive integers appearing in the first i terms of the sequence that contain p (but not q) and q (but not p) in their supports (respectively.) Define $P(i) := |P_i|$ and $Q(i) := |Q_i|$ to be the sizes of these sets. Likewise, let $PQ(i) := |PQ_i|$ denote the number of distinct positive integers appearing in the first i terms of the sequence which contain both p and q in their supports.

The pattern (2.1) implies that after a finite number of terms, at least half of the new 1s in the truth table of char_k contribute to PQ_i (namely, all of the 1s that occur in position (B)) and the rest of the new 1s (which occur in position (C)) contribute to P_i or Q_i. Because every power of two must appear infinitely many times by Theorem 27, it must be that there exists i' such that $i > i'$ implies $PQ(i) > P(i)$ or $PQ(i) > Q(i)$. In other words, the function $PQ(i)$ grows faster than at least one of $P(i)$ or $Q(i)$.

On the other hand, every time pattern 2.1 occurs, both $m_1 + 2^p$ and $m_1 + 2^q$ are smaller candidates than $m_1 + k$ for the term in position (B). If the smallest unused candidate for position (B) happens to be $m_1 + k$, then it must be the case that both $m_1 + 2^p$ and $m_1 + 2^q$ have already occurred in the sequence. This implies there exists i' such that $i > i'$ implies $Q(i) \geq PQ(i)$ and $P(i) \geq PQ(i)$, which is a contradiction of the other lower bound for $PQ(i)$ from the previous paragraph. Hence, our original assumption that property (iii) for k to be a candidate for the $(j+2)$nd term holds only a finite number of times after an occurrence of 01 in the truth table of char_k is false. By Theorem 23, this implies that k must occur in the sequence. □

Theorem 29. *Let $k = 2^p + 2^q$ be a positive integer with $\omega_2(k) = 2$. Then, there is no positive integer i' such that $i > i'$ implies $\mathrm{char}_k(i) = 1$.*

Proof. Suppose that there exists i' such that $i > i'$ implies $\mathrm{char}_k(i) = 1$. Using Iverson bracket notation, define a function $f : \mathbb{N}_{>i'} \to \{0,1\} \times \{0,1\}$ as

$$f(m) := (\Big[p \in \mathrm{supp}_2\ a(m)\Big], \Big[q \in \mathrm{supp}_2\ a(m)\Big]) \tag{2.2}$$

In the sequence $(f(i))_{i \geq i'}$, there cannot be any occurrence of $(0,0)$ because $f(j) = (0,0)$ implies $\mathrm{char}_k(j) = 0$. There also cannot be any occurrences of $(1,1)$ because $f(j) = (1,1)$ implies $f(j+2) = (0,0)$. The patterns $(1,0),(0,1),(1,0)$ and $(0,1),(1,0),(0,1)$ also cannot occur in the sequence $(f(i))_{i \geq i'}$ because they violate property (ii) of Theorem 24. So, the only possibility is that $(f(i))_{j \geq i'}$ follows the pattern $(1,0),(1,0),(0,1),(0,1)$ modulo 4 (after some suitable offset.)

Recall that whenever a new power of two 2^a appears in the sequence for the first time, it must occur in a number of the form $2^a + 2^b$. If a new power of two were introduced in position 0 or position 2 of the pattern $(1,0),(1,0),(0,1),(0,1)$, it would be impossible for the support of the term where the new power of two is introduced to intersect with the support of the preceding term. Thus, new powers of two can only be introduced in positions 1 or 3. Here is a diagram illustrating what happens when a new power of two B is introduced in position 1 (a blank indicates that power of two is not specified, a 1 indicates that column's power of two is present in the support, a 0 indicates it is absent:)

$$\begin{array}{l} \quad pq \quad \cdots \quad B \\ 0:10 \\ 1:10000\cdots00001 \\ 2:01 \qquad\qquad\quad 1 \\ 3:01 \end{array}$$

Let m be any positive integer not appearing in the first i' terms of the sequence such that $\omega_2(m) \geq 2$ and $\text{supp}_2(m) \cap \text{supp}_2(k) = \emptyset$. Out of the infinitely many terms where a new power of two 2^B is introduced, it is almost always the case that $B \notin \text{supp}_2(m)$. This implies that there are an infinite number of 0s in the truth table of char_m. By Theorem 27, the number of 1s occurring in the truth table of char_m is also infinite. Thus, there are an infinite number of occurrences of 01 in the truth table of char_m. By Theorem 28, this implies that m appears in the sequence at some point after the i'th term. Since $\text{supp}_2(m)$ is disjoint from $\text{supp}_2(k)$, this is a contradiction of the assumption that $\text{char}_k(i) = 1$ for all $i > i'$. $\square$

Theorem 30. *Every positive integer n with $\omega_2(n) \geq 2$ appears in the sequence.*

Proof. Let k be a positive integer with $\omega_2(k) = 2$. An implication of Theorem 27 is that there is no positive integer i' such that $i > i'$ implies $\text{char}_k(i) = 0$. An implication of Theorem 29 is that there is no i' such that $i > i'$ implies $\text{char}_k(i) = 1$. This proves that there are infinite occurrences of 01 in the truth table of char_k. By Theorem 28, the term k must appear in the sequence.

By the previous paragraph, for any positive integer n with $\omega_2(n) \geq 2$ there are infinitely many positive integers x with $\omega_2(x) = 2$ and $\text{supp}_2(x) \cap \text{supp}_2(n) \neq \emptyset$ appearing in the sequence. There are also infinitely many positive integers y with $\omega_2(y) = 2$ and $\text{supp}_2(y) \cap \text{supp}_2(n) = \emptyset$ appearing in the sequence. This implies that there are infinitely many occurrences of 01 in the truth table of $\text{char}_n(i)$. By Theorem 28, n must appear in the sequence. $\square$

CHAPTER 3
APPLICATIONS TO THE ENOTS WOLLEY SEQUENCE

In Chapter 2, the analog of Conjecture 14 for the binary Enots Wolley sequence was proved. The purpose of this chapter is to present the result of efforts to apply the techniques of Chapter 2 to the number-theoretic Enots Wolley sequence. Namely, the climax of this chapter is Theorem 34 which is the number-theoretic analog of Theorem 27.

Throughout this chapter, $a(n)$ shall always denote the number-theoretic Enots Wolley sequence of Definition 13 and p_n will denote the nth prime number.

3.1 Basic properties

Before it is possible to prove Theorem 34, it is necessary to establish that the basic results of Section 2.1 also hold for the number-theoretic analog. The following is the number-theoretic analog of Theorem 24, and the proof is virtually the same as in the binary case:

Theorem 31. *For $n > 2$, $a(n)$ is the smallest number m not yet in the sequence such that:*

i) $\operatorname{supp}(m) \cap \operatorname{supp}(a(n-1))$ *is nonempty,*

ii) $\operatorname{supp}(m) \cap \operatorname{supp}(a(n-2))$ *is empty, and*

iii) the set $\operatorname{supp}(m) \setminus \operatorname{supp}(a(n-1))$ *is nonempty.*

Proof. The number-theoretic Enots Wolley sequence is defined in Definition 13 as the LES satisfying some property P. It is immediate that the number-theoretic Enots Wolley sequence exists because 1 produces a partial sequence $(a_i)_{1 \le i \le n}$ that satisfies P for all n. This proves that the number-theoretic Enots Wolley sequence is well-defined.

It remains to be shown that it satisfies properties (i), (ii) and (iii). Conditions (i) and (ii) are already part of Definition 13. Condition (iii) is a consequence of (i) and (ii) because if (iii) fails for some term $a(n)$, there is no way for property (ii) to hold for the term $a(n+1)$. This proves all candidates for any term $a(n)$ must satisfy (i)-(iii).

Finally, it is necessary to prove that all numbers satisfying properties (i)-(iii) for the term $a(n)$ are candidates. In other words, it must be shown there is no need for any additional rules to guarantee that Algorithm 2 does not lead to any dead ends. For this, observe that there will always be some way to continue the sequence given that the previous two terms satisfy (i)-(iii). □

Analogous to the binary Enots Wolley sequence, an implication of Theorem 31 is that only numbers n with $\omega(n) \geq 2$ can appear in the sequence. As the following theorem demonstrates, this lower bound is met when a digit appears in a term of the sequence for the first time:

Theorem 32. *If n is the least positive integer such that $v \in \operatorname{supp}(a(n))$, the term $a(n)$ satisfies $\omega(a(n)) = 2$.*

Proof. The proof is virtually the same as the proof of Theorem 25. □

Theorem 32 justifies the following definition:

Definition 33. For distinct primes p, q, if $a(n) = pq$ is the first term of the sequence such that $p \in \operatorname{supp}(a(n))$, say that p is *introduced* by q.

3.2 New results

The only theorem appearing in this section is the number theoretic analog of Theorem 27. Unlike the previous results on the number-theoretic Enots Wolley sequence, the proof of Theorem 34 is non-trivially different from its binary analog.

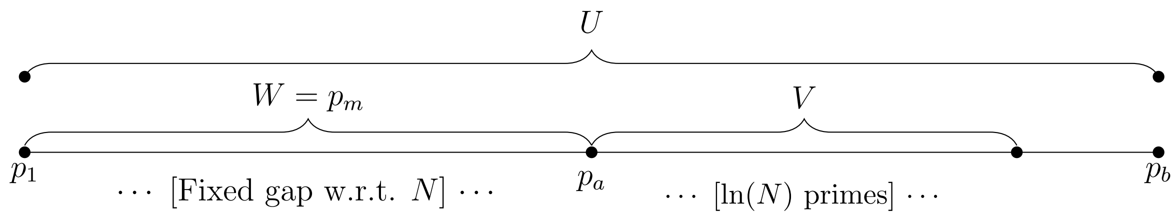

Figure 3.1: Visualization of the variables used in the proof of Theorem 34. Curly braces indicate the possible range of the corresponding prime.

Here is a brief explanation of the difficulty in adapting Theorem 27 to the number-theoretic analog. While it is easy to determine the larger of two numbers by comparing their binary representations, it is more difficult to order two numbers by looking at their prime factorizations. To overcome the problems arising from this, it will be necessary to invoke the prime number theorem and do some elementary analysis.

Theorem 34. *For all primes p_m, the truth table of* char_{p_m} *has infinitely many occurrences of* 01*.*

Proof. Since the truth table of a single prime cannot have any occurrences of 111, it must only be shown that there is no i' such that $i > i'$ implies $\mathrm{char}_{p_m}(i) = 0$. Assume that such an i' does exist. The goal is to prove that this assumption leads to a contradiction; namely, that there are further occurrences of 1 in the truth table of p_m after the i'-th position.

Let p_a be the largest prime that has occurred in the sequence before the (i')th term. Let p_b be a sufficiently large prime so that $b - a > N$ where $N = N(p_m)$ is some number depending on p_m. It will be shown that for any p_m, it is possible to choose $N(p_m)$ sufficiently large so that p_m occurs in the sequence at some point after the i'th term, contradicting the assumption that $\mathrm{char}_{p_m}(i) = 0$ for all $i > i'$.

Let n be the least positive integer such that $p_b \in \mathrm{supp}(a(n))$. It is now necessary to make the following three definitions:

- $U :=$ The prime that introduces p_b in the nth term.
- $V :=$ A prime distinct from U such that $p_a < V \leq p_{a+\lfloor \ln(N) \rfloor}$.
- $W := p_m$.

It will be shown that unless all of the $\lfloor \ln(N) \rfloor$ choices of V are present in $a(n-2)$, the number UVW is a smaller unused candidate for $a(n)$ than Up_b.

First, consider the case where there is some possible choice of V that does not violate property (ii). It must be shown that $UVW < p_b U$. By dividing both sides of this inequality by U, observe that this is equivalent to proving $VW < p_b$. Since p_m is a prime smaller than the prime p_a, it suffices to prove that $Vp_a < p_b$. Furthermore, because the largest possible value of V is $p_{a+\lfloor \ln(N) \rfloor}$, it suffices to prove

$$p_{a+\lfloor \ln(N) \rfloor} p_a < p_b \tag{3.1}$$

Equivalently,

$$p_{a+\lfloor \ln(N) \rfloor} p_a < p_{N+a} \tag{3.2}$$

where a is a constant that does not depend on N. For this, recall that the prime number theorem states that $p_n \sim n \ln(n)$, which means

$$\lim_{n \to \infty} \frac{p_n}{n \ln(n)} = 1 \tag{3.3}$$

Applying the prime number theorem to each side of Equation 3.2, one obtains

$$\begin{aligned} p_{a+\lfloor \ln(N) \rfloor} p_a &\sim \left(a + \lfloor \ln(N) \rfloor\right) \ln\left(a + \lfloor \ln(N) \rfloor\right) \left(a \ln(a)\right) \\ p_{N+a} &\sim (N+a) \ln(N+a) \end{aligned} \tag{3.4}$$

By elementary analysis, it is clear that the left hand side of Equation 3.2 is dominated by the right hand side. Thus, it is possible to choose N sufficiently large so that Equation 3.2 holds.

Now, it has been established that it is possible to choose N such that $UVW < p_b U$. It must also be shown that UVW does not appear in the sequence before the nth term, and that UVW satisfies the properties of Theorem 31 to be a candidate for the term $a(n)$. These parts of this claim follow from the following properties of the three factors U, V and W:

- U: Ensures property (i) for UVW to be a candidate for the nth term holds.
- V: Because W has never occurred together with V, this ensures UVW has not appeared in the first $n-1$ terms of the sequence.
- W: Ensures that if $a(n) = UVW$, then $\text{char}_p(n) = 1$ (contradicting the assumption that $\text{char}_p(n) = 0$), and that property (iii) holds for UVW to be a candidate for $a(n)$.

This proves that as long as V isn't barred from appearing in the term $a(n)$ by property (ii), the number UVW is a smaller candidate for $a(n)$ that does not appear in the first $n-1$ terms of the sequence. In the case that all possible choices of V are present in $a(n-2)$, it is possible to construct an alternative value of $a(n-2)$ that produces a lexicographically earlier sequence as follows:

The number of primes that can serve as a valid choice of V is $\lfloor \ln(N) \rfloor$. Hence, if all possible choices of V are present in $a(n-2)$, then it must be true that $\omega(a(n-2)) \geq \lfloor \ln(N) \rfloor$. Because the quantity $\lfloor \ln(N) \rfloor$ is arbitrarily large, so must be the term $a(n-2)$. Since none of the factors that are possible choices for V have yet appeared in the sequence in a term together with p_m, it is possible to erase one of the factors that are possible choices for V and replace it with p_m in $a(n-2)$ to obtain a smaller unused candidate for the $(n-2)$nd term. □

It is possible that other theorems from Chapter 2 can be adapted to the number-theoretic Enots Wolley sequence as well, but the details are far from obvious. Several attempts at this by this author have failed. It seems that proving Conjecture 14 would require a substantial portion of additional ingenuity beyond anything that has been tried so far.

It is likely that there are many Conjectures like 14 that as yet still unknown to mathematics. For example, there are entire analog EKG families that have not yet even been written down. Study of these more obscure EKG-like sequences could perhaps indirectly lead to progress towards Conjecture 14 and conjectures like it.

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