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Stochastic Stability of the Continuous-Time Extended Kalman Filter

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Abstract: The error behaviour of the extended Kalman filter is analysed. It is proved that the estimation error remains bounded if the system satisfies a detectability condition and both the initial estimation error and the disturbing noise terms are small enough. Moreover, some selected cases with both bounded and unbounded estimation error are demonstrated by numerical simulations.

1 Introduction

Kalman–Bucy filtering (see e.g. [1, 7, 13, 15]) is an important and widely used tool for the state and parameter estimation of stochastic systems. Although it was originally developed as an optimal filter for linear systems, an application to nonlinear systems is also possible. The usual procedure is to linearise the nonlinear system at the current estimate, leading to the extended Kalman filter [1, 7, 13, 15]. This technique has turned out to be one of the most useful methods for the state and parameter estimation of nonlinear stochastic systems (cf. [4, 9]).

Unfortunately, its superior practical usefulness is accompanied with a heuristic theoretical derivation, which makes a mathematically rigorous derivation of the extended Kalman filter difficult. Moreover the stability and convergence properties of the continuous-time extended Kalman filter are hard to analyse and have been developed only for special applications: in [16, 30, 31] the extended Kalman filter is examined if it is used as a parameter estimator for linear systems; in [5] an extended Kalman filter based on nonlinear high-gain observer is introduced; and the properties of the extended Kalman filter employed as an observer for nonlinear deterministic systems are explored in [20, 22, 23]. Moreover, interesting relations between the observability of the nonlinear system and the existence of positive-definite solutions for the Riccati differential equations are given in [3].

Motivated by the stability of the usual Kalman–Bucy filter for linear systems (see [13], chapter 7, section 6, and [14]) and by successful application of the stochastic-stability theory to solve nonlinear-estimation problems in [28, 32] analogous results are examined for the nonlinear case, i.e. for the extended Kalman filter. In particular it is shown that the estimation error of the extended Kalman filter remains bounded if the initial estimation-error and the disturbing-noise terms are sufficiently small. To carry out the proof supermartingales are employed; this is a common approach in the stability theory of stochastic differential equations [2, 6, 8].

Throughout this paper, $\|\cdot\|$ denotes the Euclidian norm of real vectors in $\mathbb{R}^q$ or the spectral norm of real matrices, $E\{x\}$ is the mean value of x, $E\{x|y\}$ the mean value of x conditional on y, and $P\{\cdot\}$ the probability of an event. All equations and inequalities involving random variables are to be understood in the sense that they must hold with probability 1.

2 State estimation, stochastic boundedness and supermartingales

Consider a nonlinear stochastic system modelled by the Ito stochastic differential equations

$$dz(t) = f[z(t), x(t)]\,dt + G(t)\,dw(t) \tag{1}$$

$$dy(t) = h[z(t)]\,dt + D(t)\,dv(t) \tag{2}$$

where the state $z(t)$, the input $x(t)$ and the output $y(t)$ are stochastic processes with sample paths in $\mathbb{R}^q$, $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively. Furthermore $t \geq 0$ is the time, the noise terms $v(t)$, $w(t)$ are $\mathbb{R}^k$-and $\mathbb{R}^l$-valued uncorrelated standard Wiener processes, and $D(t)$, $G(t)$ are time-varying $m \times k$ and $q \times l$ matrices, respectively. The initial condition $z(0) = z_0$ is an unknown constant vector (with probability 1). The nonlinear functions f, h are assumed to be continuously differentiable and satisfy appropriate conditions, such that the differential equations eqns. 1 and 2 have unique solutions in the proper stochastic sense (see e.g. [8], chapter 2, section 6, p. 37). For this nonlinear system, a state estimator is introduced given by

$$d\hat{z}(t) = f[\hat{z}(t), x(t)]\,dt + K(t)[dy(t) - h(\hat{z}(t))\,dt] \tag{3}$$

with the state estimate $\hat{z}(t)$ and the observer gain $K(t)$, which is in general a stochastic process with time varying

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$q \times m$ matrices as sample paths. The nonlinear functions f and h are expanded up to first order via

$$f[z(t), x(t)] - f[\hat{z}(t), x(t)] = A(t)[z(t) - \hat{z}(t)] + \phi[z(t), \hat{z}(t), x(t)] \tag{4}$$

and

$$h[z(t)] - h[\hat{z}(t)] = C(t)[z(t) - \hat{z}(t)] + \chi[z(t), \hat{z}(t)] \tag{5}$$

where

$$A(t) = \frac{\partial f}{\partial z}[\hat{z}(t), x(t)] \tag{6}$$

and

$$C(t) = \frac{\partial h}{\partial z}[\hat{z}(t)] \tag{7}$$

are matrix-valued stochastic processes and $\varphi[z(t), \hat{z}(t), x(t)]$ and $\chi[z(t), \hat{z}(t)]$ are the remaining nonlinear terms. The estimation error is defined by

$$\zeta(t) = z(t) - \hat{z}(t) \tag{8}$$

Subtracting eqn. 3 from eqn. 1 and using eqns. 2, 4 and 5 leads to the error evolution

$$d\zeta(t) = ([A(t) - K(t)C(t)]\zeta(t) + n(t))dt + \Gamma(t)\begin{bmatrix} dw(t) \\ dv(t) \end{bmatrix} \tag{9}$$

with

$$n(t) = \varphi[z(t), \hat{z}(t), x(t)] - K(t)\chi[z(t), \hat{z}(t)], \tag{10}$$

$$\Gamma(t) = [G(t) - K(t)D(t)] \tag{11}$$

To examine the dynamics of the estimation error the following two concepts of boundedness for solutions of stochastic differential equations are used in this paper (cf. [6, 17, 28, 33]):

Definition 2.1: The stochastic process $\zeta(t)$ is said to be stochastically sample path bounded, if for every $\alpha > 0$ there is a $\beta(\alpha) > 0$ such that

$$P\{\sup_{t \geq 0} \|\zeta(t)\| \leq \beta(\alpha)\} \geq 1 - \alpha_j \tag{12}$$

Definition 2.2: The stochastic process $\zeta(t)$ is said to be exponentially bounded in mean square if there are real numbers η, ϑ, $\nu > 0$ such that

$$E\{\|\zeta(t)\|^2\} \leq \eta\|\zeta(0)\|^2 \exp(-\vartheta t) + \nu \tag{13}$$

holds for every $t \geq 0$.

A commonly used tool to prove stochastic boundedness or stability are supermartingales (see e.g. [6], section 2.3, p. 49). They play a similar role in the stability theory of stochastic differential equations as the Lyapunov functions do for deterministic differential equations; in some sense they are the stochastic analogue for the Lyapunov functions (see e.g. [2], section 11.2, p. 189). For the application in Section 3 standard results about stochastic boundedness are adapted to the specific requirements which arise in the analysis of the error dynamics (eqn. 9) for the extended Kalman filter. For this purpose recall the definition of the differential generator (see e.g. [2, 6, 8]) and adapt it to the nonlinear-estimation problem [32]: Under conditions guaranteeing the existence and uniqueness of the solutions, the following stochastic system is considered:

$$d\zeta(t) = \tilde{f}[\zeta(t), t]dt + \tilde{G}(t)d\tilde{w}(t) \tag{14}$$

For a given stochastic process $V[\zeta(t), t]$ the differential generator can be defined by

$$\begin{aligned} \mathcal{L}V(\zeta, t) = {} & \frac{\partial V}{\partial t}(\zeta, t) + \frac{\partial V}{\partial \zeta}(\zeta, t)\tilde{f}(\zeta, t) \\ & + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta, t)[\tilde{G}(t)\tilde{G}^T(t)]_{i,j} \end{aligned} \tag{15}$$

with $\zeta = [\zeta_1, \ldots, \zeta_q]^T$ and $[\tilde{G}(t)\tilde{G}^T(t)]_{i,j}$ denoting the matrix element of $\tilde{G}(t)\tilde{G}^T(t)$ in the ith row and the jth column. Comparing eqns. 9 and 14, and using eqn. 15 yields

$$\begin{aligned} \mathcal{L}V(\zeta, t) = {} & \frac{\partial V}{\partial t}(\zeta, t) + \frac{\partial V}{\partial \zeta}(\zeta, t)([A(t) - K(t)C(t)]\zeta(t) + n(t)) \\ & + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta, t)[\Gamma(t)\Gamma^T(t)]_{i,j} \end{aligned} \tag{16}$$

with $[\Gamma(t)\Gamma^T(t)]_{i,j}$ denoting the matrix element of $\Gamma(t)\Gamma^T(t)$ in the ith row and the jth column.

Furthermore, the sum in eqn. 16 can be written (see e.g. [2], section 5.3, p. 104) as

$$\begin{aligned} & \sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta, t)[\Gamma(t)\Gamma^T(t)]_{i,j} \\ & = \mathrm{tr}(\Gamma(t)\Gamma^T(t)\mathrm{Hess}[V(\zeta, t)]) \\ & = \mathrm{tr}(\mathrm{Hess}[V(\zeta, t)]\Gamma(t)\Gamma^T(t)) \end{aligned} \tag{17}$$

where $\mathrm{Hess}(\cdot)$ denotes the Hessian matrix.

Lemma 2.3: Assume there is a stochastic process $V[\zeta(t), t]$ and real numbers $\underline{v}$, $\bar{v}$, γ, $\mu > 0$ such that

$$\underline{v}\|\zeta(t)\|^2 \leq V[\zeta(t), t] \leq \bar{v}\|\zeta(t)\|^2 \tag{18}$$

and

$$\mathcal{L}V[\zeta(t), t] \leq -\gamma V[\zeta(t), t] + \mu \tag{19}$$

are fulfilled. Then the stochastic process $\zeta(t)$ is exponentially bounded in mean square, i.e.

$$E\{\|\zeta(t)\|^2\} \leq \frac{\bar{v}}{\underline{v}}\|\zeta(0)\|^2 \exp(-\gamma t) + \frac{\mu}{\underline{v}\gamma} \tag{20}$$

for every $t \geq 0$. Moreover the stochastic process $\zeta(t)$ is sample-path bounded.

Proof: This lemma contains a combination of theorem 1 in [33] and theorem 5.2, p. 129 in [6]. See also [8], part II, chapter 3, section 13, p. 319, proposition 2 as well as [17, 19, 28].

3 Error bounds for the extended Kalman filter

Because the extended Kalman filter has been developed from an application point of view rather than mathematical rigour, most treatments of the extended Kalman filter use a simplified notation instead of the Ito stochastic calculus by adding 'white noise' terms to ordinary differential equations (see, e.g., [7], section 6.1, p. 188, table 6.1-2, or [15], section 5.3, p. 265, table 5.3-2). However, for the present purpose a mathematically accurate formulation using Ito stochastic differential equations is more appropriate. Nevertheless an attempt is made to follow the common engineering literature [1, 7, 13, 15] as closely as possible.

Definition 3.1: An extended Kalman filter is given by the following equations: Differential equation for the state estimate:

$$d\hat{z}(t) = f[\hat{z}(t), x(t)]dt + K(t)(dy(t) - h[\hat{z}(t)]dt) \quad (21)$$

Riccati differential equation:

$$dP(t) = [A(t)P(t) + P(t)A^T(t) + Q(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t)]dt \quad (22)$$

Linearisation:

$$A(t) = \frac{\partial f}{\partial z}[\hat{z}(t), x(t)] \quad (23)$$

$$C(t) = \frac{\partial h}{\partial z}[\hat{z}(t)] \quad (24)$$

Kalman gain:

$$K(t) = P(t)C^T(t)R^{-1}(t) \quad (25)$$

The output $y(t)$ of the system to be observed is given by eqns. 1 and 2, $Q(t)$ is a time-varying symmetric positive-definite $q \times q$ matrix and $R(t)$ a time-varying positive-definite $m \times m$ matrix.

Remark: A usual choice for the matrices $Q(t)$ and $R(t)$ are the covariances for the corrupting noise terms in eqns. 1 and 2, i.e.

$$Q(t) = G(t)G^T(t) \quad (26)$$

$$R(t) = D(t)D^T(t) \quad (27)$$

This case corresponds also to a deterministic maximum-likelihood estimation according to [18]. However, this is not the only possibility. Especially for estimation problems with zero noise, i.e. for

$$G(t)G^T(t) = 0 \quad (28)$$

$$D(t)D^T(t) = 0 \quad (29)$$

or for systems with severe nonlinearities, supplementary alternatives are of particular interest [20, 21, 23].

With these prerequisites it is possible to state the following result, where the matrix inequalities $\Omega \leq \Delta$ and further $\Omega - \Delta \leq 0$ mean that the matrix $\Omega - \Delta$ is negative semidefinite.

Theorem 3.2: Consider a nonlinear stochastic system with state differential equations (eqns. 1 and 2) and an extended Kalman filter as in Definition 3.1. Let the following assumptions hold:

(i) There are real numbers $\bar{c}, \underline{p}, \bar{p}, \underline{q}, \underline{r} > 0$ such that the following bounds are satisfied for every $t \leq 0$:

$$\|C(t)\| \leq \bar{c} \quad (30)$$

$$\underline{p}I \leq P(t) \leq \bar{p}I \quad (31)$$

$$\underline{q}I \leq Q(t) \quad (32)$$

$$\underline{r}I \leq R(t) \quad (33)$$

(ii) There are real numbers $\varepsilon_\phi, \varepsilon_\chi, \kappa_\phi, \kappa_\chi > 0$ such that the nonlinear functions ϕ, χ in eqn. 10 are bounded by

$$\|\phi(z, \hat{z}, x)\| \leq \kappa_\phi \|z - \hat{z}\|^2 \quad (34)$$

$$\|\chi(z, \hat{z}\| \leq \kappa_\chi \|z - \hat{z}\|^2 \quad (35)$$

for $z, \hat{z} \in \mathbb{R}^q$ and $x \in \mathbb{R}^p$ with $|z - \hat{z}| \leq \varepsilon_\phi$ and $|z - \hat{z}| \leq \varepsilon_\chi$, respectively.

Then there exist real numbers, $\delta, \varepsilon > 0$ such that the estimation error $\zeta(t)$ given by eqn. 8 is exponentially bounded in mean square and stochastically sample-path bounded, if the initial estimation error satisfies

$$\|\zeta(0)\| \leq \varepsilon \quad (36)$$

and the covariance matrices of the noise terms are bounded via

$$G(t)G^T(t) \leq \delta I \quad (37)$$

$$D(t)D^T(t) \leq \delta I \quad (38)$$

for every $t \geq 0$.

3.1 Remarks

(i) For linear systems the matrix $P(t)$ is the covariance matrix for the estimation error and therefore this theorem is trivially true in this case.
(ii) Eqn. 31 in assumption 1 is closely related to the observability and detectability properties of the system. These relations are discussed in Section 4.
(iii) To obtain the error bounds, the matrices $Q(t)$ and $R(t)$ need not be the covariances of the noise terms. Any other positive-definite matrices can be chosen as well.
(iv) To obtain κ_ϕ and κ_χ in eqns. 34 and 35, a compact subset $\mathcal{K}$ of $\mathbb{R}^q$ was considered. The bounds defined by eqns. 34 and 35 for $z, \hat{z} \in \mathcal{K}$ can be calculated by a standard estimation via an integral formula (see e.g. [12], chapter 20, section 168, or [29], section 8.1.3). Let f_i, h_i be the components of f and h, respectively. If f, h are twice differentiable (with respect to z) for every $z \in \mathcal{K}$, it follows that the Hessian matrices of f_i and h_i are bounded with respect to the Euclidian norm of matrices. The constants κ_ϕ, κ_χ are then given by

$$\kappa_\phi = \max_{1 \leq i \leq q} \sup_{z \in \mathcal{K}} \|\text{Hess } f_i(z, x)\| \quad (39)$$

$$\kappa_\chi = \max_{1 \leq i \leq m} \sup_{z \in \mathcal{K}} \|\text{Hess } h_i(z)\| \quad (40)$$

(v) Under unfortunate conditions (e.g. severe nonlinearities), the constants ε, δ may be very small. This may cause a restrictive applicability of this theorem under these conditions. See also the last paragraph in Section 5.
(vi) Regarding the restrictiveness of condition (eqn. 30), note that, for many applications the state variables (which often represent physical quantities) are bounded inside reasonable limits. Outside this 'operating area' the form of the function h has no influence on the behaviour of the system. If the function h satisfies $|(\partial h/\partial z)(z)| \leq \bar{c}$ for every physical reasonable value of the state vector z, it can be assumed without loss of generality that $|C(t)| \leq \bar{c}$ holds.
(vii) Slight modifications of the Riccati differential equation (eqn. 22) may lead to an improvement of the dynamical behavior of the estimation error $\zeta(t)$, especially better tolerance relating to large initial errors (see e.g. [10] or [25]).

To prove the boundedness of the estimation error use is made of the following lemma.

Lemma 3.3: Consider real vectors $z, \hat{z} \in \mathbb{R}^q$, $x \in \mathbb{R}^p$, a symmetric positive-definite $q \times q$ matrix P, a positive-definite $m \times m$ matrix R, a $m \times q$ matrix C and nonlinear functions ϕ and χ, such that the following assumptions hold:

(i) There are positive real numbers $\bar{c}, p, r > 0$ such that the considered matrices fulfill the bounds

$$\|C\| \le \bar{c} \tag{41}$$

$$\underline{p}I \le P \tag{42}$$

$$\underline{r}I \le R \tag{43}$$

(ii) There are positive real numbers ϵ_ϕ, ϵ_χ, κ_ϕ, $\kappa_\chi > 0$ such that

$$\|\phi(z, \hat{z}, x)\| \le \kappa_\phi \|z - \hat{z}\|^2 \tag{44}$$

$$\|\chi(z, \hat{z})\| \le \kappa_\chi \|_z - \hat{z}\|^2 \tag{45}$$

hold for $|z - \hat{z}| \le \epsilon_\phi$ and for $|z - \hat{z}| \le \epsilon_\chi$, respectively.

Let n be given by

$$n = \phi(z, \hat{z}, x) - K\chi(z, \hat{z}) \tag{46}$$

$\Pi = P^{-1}$ and $K = PC^T R^{-1}$. Then there is a positive real number $\kappa_{nonl} > 0$ such that

$$2(z - \hat{z})^T \Pi n \le \kappa_{nonl} \|z - \hat{z}\|^3 \tag{47}$$

holds for $|z - \hat{z}| \le \epsilon'$ with $\epsilon' = \min(\epsilon_\phi, \epsilon_\chi)$.

Proof: Applying the triangle inequality and using $K = PC^T R^{-1}$, $\Pi P = I$ the left-hand side of eqn. 47 yields.

$$\begin{aligned} \|^2(z - \hat{z})^T \Pi n\| &= \|2(z - \hat{z})^T \Pi \phi(z, \hat{z}, x) \\ &\quad - 2(z - \hat{z})^T \Pi K_\chi(z, \hat{z})\| \\ &\le \|2(z - \hat{z})^T \Pi \phi(z, \hat{z}, x)\| \\ &\quad + \|2(z - \hat{z})^T C^T R^{-1} \chi(z, \hat{z})\| \end{aligned} \tag{48}$$

Set $\epsilon' = \min(\epsilon_\phi, \epsilon_\chi)$. Using eqns. 41–45 and $\Pi = P^{-1}$ one obtains

$$\begin{aligned} \|2(z - \hat{z})^T \Pi n\| &\le 2\|z - \hat{z}\| \frac{\kappa_\phi}{\underline{p}} \|z - \hat{z}\|^2 \\ &\quad + 2\|z - \hat{z}\| \frac{\bar{c}\kappa_\chi}{\underline{r}} \|z - \hat{z}\|^2 \end{aligned}$$

for $|z - \hat{z}| \le \epsilon'$. Defining

$$\kappa_{nonl} = \frac{2\kappa_\phi}{\underline{p}} + \frac{2\bar{c}\kappa_\chi}{\underline{r}} \tag{49}$$

and using (eqn. 46) the desired property (eqn. 47) is obtained.

Proof of Theorem 3.2: See Appendix (Section 9).

Remark: For later use it is necessary to establish some estimates for ϵ and δ. From the Appendix, (eqn. 88) we have immediately

$$\epsilon = \min\left(\epsilon_\phi, \epsilon_\chi, \frac{q}{2\kappa_{nonl}\bar{p}^2}\right) \tag{50}$$

with κ_{nonl} given by (eqn. 49). For the evaluation of δ use theorem 5.2 in [6]. According to this theorem it is necessary to take care that, for $\tilde{\epsilon} \le |\zeta(t)| \le \epsilon$ with some $\tilde{\epsilon}$ the inequality (cf. Appendix, Section 9, eqn. 89)

$$\mathcal{L}V[\zeta(t), t] \le -\frac{qp}{2\bar{p}^2} V[\zeta(t), t] + \kappa_{noise}\delta \le 0 \tag{51}$$

is fulfilled to guarantee the boundedness of the estimation error. Choosing

$$\delta = \frac{qp\tilde{\epsilon}^2}{2\bar{p}^3 \kappa_{noise}} \tag{52}$$

with some $\tilde{\epsilon} < \epsilon$ one has, for $|\zeta(t)| \ge \epsilon$,

$$\kappa_{noise}\delta \le \frac{qp}{2\bar{p}^3} \|\zeta(t)\|^2 \le \frac{qp}{2\bar{p}^2} V[\zeta(t), t] \tag{53}$$

i.e. (eqn. 51) holds.

4 Relations to uniform detectability

To prove the boundedness of the estimation error bounds are required for the solution $P(t)$ of the Riccati differential equation (eqn. 32) according to eqn. 31. This condition is closely related to observability and detectability properties of the system to be observed. The classical treatment for linear systems with deterministic state matrices has been given in [14], and generalisations to stochastic state matrices have been proposed in [31]. Nonlinear stochastic systems are treated in [3], where the following detectability notation has been proposed.

Definition 4.1: The pair

$$\left[\frac{\partial f}{\partial z}(z, x), \frac{\partial b}{\partial z}(z)\right] \quad z \in \mathbb{R}^q, x \in \mathbb{R}^p \tag{54}$$

is called uniformly detectable if there is a bounded matrix valued function $\Lambda(z)$ and a real number $\gamma > 0$ such that

$$w^T \left[\frac{\partial f}{\partial z}(z, x) + \Lambda(z) \frac{\partial h}{\partial z}(z)\right] w \le -\gamma \|w\|^2 \tag{55}$$

holds for every w, $z \in \mathbb{R}^q$ and $x \in \mathbb{R}^p$.

Now it is possible to state the following result.

Theorem 4.2: Consider a nonlinear stochastic system with state differential equations eqns 1 and 2 and an extended Kalman filter as in definition 3.1. Let the following assumptions hold:

(i) There are positive real numbers $\bar{c}$, q, $r > 0$ such that the following bounds are satisfied for every $t \ge 0$:

$$\|C(t)\| \le \bar{c} \tag{56}$$

$$\underline{q}I \le Q(t) \tag{57}$$

$$\underline{r}I \le R(t) \tag{58}$$

(ii) The pair

$$\left[\frac{\partial f}{\partial z}(z, x), \frac{\partial h}{\partial z}(z)\right], z \in \mathbb{R}^q, x \in \mathbb{R}^p \tag{59}$$

is uniformly detectable according to definition 4.1.

(iii) There are real numbers ϵ_ϕ, ϵ_χ, κ_ϕ, $\kappa_\chi > 0$ such that the nonlinear functions ϕ, χ in (eqn. 10) are bounded by

$$\|\phi(z, \hat{z}, x)\| \le \kappa_\phi \|z - \bar{z}\|^2 \tag{60}$$

$$\|\chi(z, \bar{z})\| \le \kappa_\chi \|z - \bar{z}\|^2 \tag{61}$$

for z, $\hat{z} \in \mathbb{R}^q$ and $x \in \mathbb{R}^p$ with $\|z - \hat{z}\| \le \epsilon_\phi$ and $\|z - \hat{z}\| \le \epsilon_\chi$, respectively

Then there exist real numbers δ, $\epsilon > 0$ such that the estimation error $\zeta(t)$ given by eqn. 8 is exponentially bounded in mean square and stochastically sample-path bounded, if the initial estimation error satisfies

$$\|\zeta(0)\| \le \epsilon \tag{62}$$

and the covariance matrices of the noise terms are bounded via

$$G(t)G^T(t) \le \delta I \tag{63}$$

$$D(t)D^T(t) \le \delta I \tag{64}$$

for every $t \ge 0$.

For the proof of this theorem use is made of the following auxiliary result.

Lemma 4.3: Assume that the pair

$$\left[\frac{\partial f}{\partial z}(z,x), \frac{\partial h}{\partial z}(z)\right], \quad z \in \mathbb{R}^q, \quad x \in \mathbb{R}^p \tag{65}$$

is uniformly detectable according to definition 4.1. Then the solution $P(t)$ of the Riccati differential equation (eqn. 22) satisfies the bound

$$\underline{p}I \leq P(t) \leq \bar{p}I \tag{66}$$

Proof: See [3], theorem 7.

Proof of Theorem 4.2: According to lemma 4.3 the solution $P(t)$ of the Riccati differential equation satisfies the bounds of eqn. 66. Therefore all requirements are satisfied to apply theorem 3.2, and the boundedness of the estimation error follows under the stated conditions. □

5 Numerical simulations

In Sections 3 and 4 it has been shown that, under certain conditions the estimation error for the extended Kalman filter remains bounded. To obtain the error bounds one requires especially a sufficiently small initial estimation error and sufficiently small noise. In this Section numerical simulations are presented, which indicate that the estimation error is bounded for small initial estimation errors and small noise and divergent for large initial estimation errors or large noise. For this purpose consider a nonlinear stochastic example system with the state differential equations eqns. 1 and 2, where

$$f(z(t), x(t)) = \begin{bmatrix} z_2(t) \\ -z_1(t) + (z_1^2(t) + z_2^2(t) - 1)z_2(t) \end{bmatrix} \tag{67}$$

and

$$h[z(t)] = \exp[c - z_2(t)] \tag{68}$$

From eqns. 67 and 68 one computes

$$\frac{\partial f}{\partial z}(z,x) = \begin{bmatrix} 0 & 1 \\ -1 + 2z_1 z_2 & z_1^2 + 3z_2^2 - 1 \end{bmatrix} \tag{69}$$

and

$$\frac{\partial h}{\partial z}(z) = \begin{bmatrix} 0 & -\exp(-z_2) \end{bmatrix} \tag{70}$$

It can be checked that the matrices fulfill the uniform-detectability condition of definition 4.1 with

$$\Lambda(z) = \begin{bmatrix} 2z_1 z_2 \exp(z_2) + 1 \\ \exp(z_2)(z_1^2 + 3z_2^2 - 1) + 1 \end{bmatrix}$$

Therefore, according to lemma 4.3 the Riccati differential equation (eqn. 22) has a bounded solution.

For a numerical solution of the stochastic differential equations eqns. 1, 2, 21 and 22, the stochastic version of Heun's method (see e.g. [6], Section 7.2, p. 192 or [26]) is employed. It is well known that this method converges to the Stratonovich solutions rather than to the Ito solutions of the considered stochastic differential equations. Therefore an appropriate correction term is added according to [2] (section 10.2, p. 178) or [6] (section 3.4, p. 95). For the numerical simulations, one case with bounded estimation error and two cases with divergent estimation error were

Table 1: Initial values and noise-weighting matrices for the numerical simulations

	Small initial error and small noise	Large noise	Large initial error
$\hat{z}(0)$	$[0.5\ 0.5]^T$	$[0.5\ 0.5]^T$	$[1.5\ 1.5]^T$
$G(t)$	$0.1I$	$0.1I$	$0.1I$
$D(t)$	0.1	$\sqrt{2}$	0.1
Error behaviour	Bounded	Divergent	Divergent
Figures	1a, 2a	1b, 2b	1c, 2c

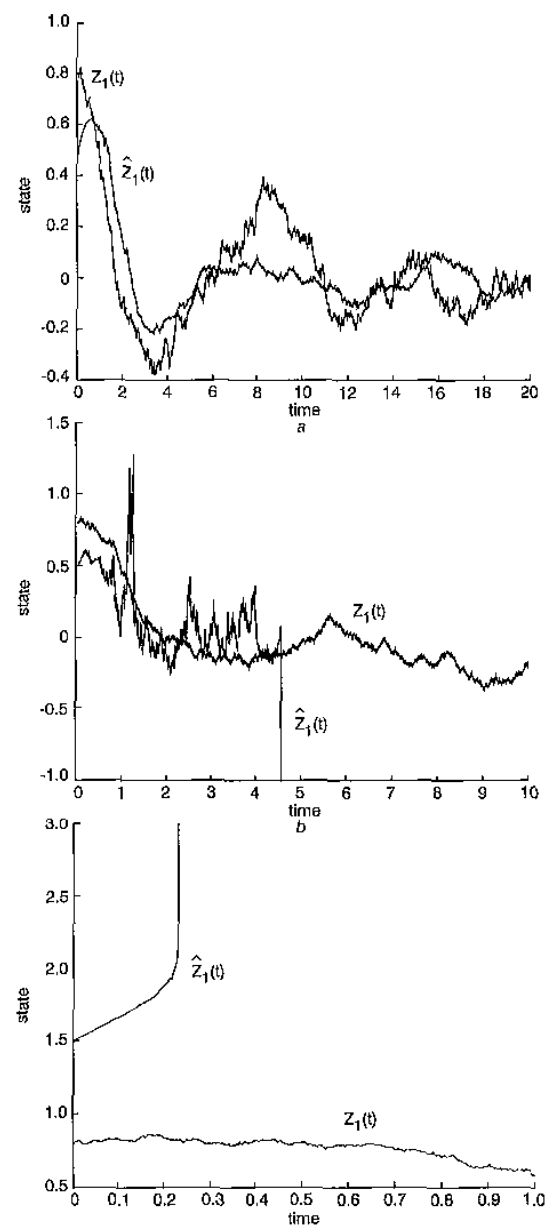

Fig. 1 *Numerical simulations of the extended Kalman filter for the example system*

a Small initial error and small noise
b Large noise
c Large initial errorz_1
$z(t)$ = state component
$\hat{z}_1(t)$ = estimate

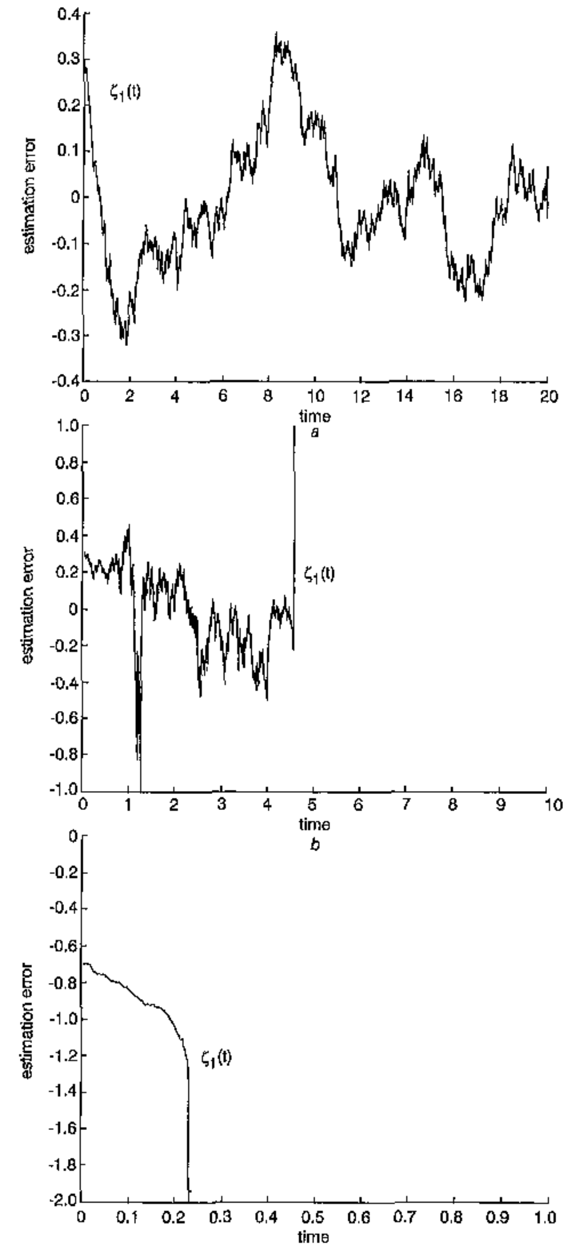

Fig. 2 *Numerical simulations of the extended Kalman filter for the example system*

a Small initial error and small noise
b Large noise
c Large initial error
$z_1(t)$ = state component
$\hat{z}_1(t)$ = estimate

considered. For all three cases simply select (see remark after definition 3.1 remark 3 after theorem 3.2)

$$Q(t) = I \tag{71}$$

$$R(t) = 1 \tag{72}$$

$$P(0) = I \tag{73}$$

and

$$z(0) = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}^T \tag{74}$$

The stochastic differential equations considered are solved numerically (provided that the solutions exist) using the Heun discretisation with stepsize $\Delta t = 10^{-3}$. The remaining matrices $G(t)$ and $D(t)$, as well as the initial value $\hat{z}(0)$, are chosen particularly for each of the three cases and are shown in Table 1. The following cases were considered: small initial error and small noise, large noise as well as large initial error. The simulation results are depicted in Figs. 1–2, where sample paths for the unknown state $z_1(t)$ and the estimated state $\hat{z}_1(t)$, as well as for the estimation error $\zeta_1(t)$, are plotted against time t.

It can be seen in Figs. 1 and 4 that for small initial error and small noise (cf. eqns. 36–38 or 62–64, respectively) the estimation error remains bounded. However, if the initial estimation error or the disturbing noise is large (i.e. eqns. 36–38 or 61–63, respectively are violated), then the estimation error is no longer bounded, as can be verified in Figs. 1*a* and 2*a*. Because of the high nonlinearities of the example system considered, the error is divergent as can be verified in Figs. 1*b*–*c* and 2*b*–*c*.

The numerical simulations have shown that the estimation error is bounded if $\|\zeta(0)\| \leq 0.4$ and $G(t)G^T(t) \leq 0.01I$, $D(t)D^T(t) \leq 0.01$ is fulfilled. However, estimating ϵ and δ via eqns. 50 and 52 yields much smaller values for the bounds. For a considered compact subset $\mathcal{K} \subseteq \mathbb{R}^q$ with $\mathcal{K} = \{z \in \mathbb{R}^q \| \|z\| \leq 3\}$ one obtains $\epsilon \leq 1.4 \cdot 10^{-4}$ and $\delta \leq 5 \cdot 10^{-14}$, respectively. Because of its conservative character, this estimation has only a theoretical meaning.

6 Conclusions

In this paper the behaviour of the estimation error for the extended Kalman filter has been examined. It has been shown that, under certain conditions, the estimation error is bounded in mean square and stochastically sample-path bounded. This fact is embodied in the theorems 3.2 and 4.2 in Sections 3 and 4. To obtain the error bounds, a good initial guess is required, along with small noise terms, and the original nonlinear stochastic system must be uniformly detectable, from which follows a bounded solution for the Riccati differential equation. In Section 5 it has been shown by numerical simulations that, for systems with severe nonlinearities, these assumptions are, although restrictive, often necessary. The numerical simulations verify that the estimation error remains bounded for small initial errors and small noise terms, moreover they indicate that the error is divergent for large initial errors or large noise power. This paper is limited to the continuous-time case; the discrete-time case is treated in [24].

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9 Appendix

Proof of theorem 3.2
Choose

$$V(\zeta(t), t) = \zeta^T(t)\Pi(t)\zeta(t) \tag{75}$$

with $\Pi(t) = P^{-1}(t)$, which is defined with probability 1, because $P(t)$ is positive definite (with probability 1) since eqn. 31 holds. From eqn. 31 it follows that

$$\frac{1}{\bar{p}} I \le \Pi(t) \le \frac{1}{\underline{p}} I \tag{76}$$

and with eqn. 75 $V[\zeta(t), t]$ is bounded by

$$\frac{1}{\bar{p}} \|\zeta(t)\|^2 \le V[\zeta(t), t] \le \frac{1}{\underline{p}} \|\zeta(t)\|^2 \tag{77}$$

From eqns. 75, 49, 16 and 17 one can obtain

$$\begin{aligned} \mathcal{L}V[\zeta(t), t] = {} & \zeta^T(t)\frac{d\Pi}{dt}(t)\zeta(t) \\ & + \zeta^T(t)[A(t) - K(t)C(t)]^T\Pi(t)\zeta(t) \\ & + \zeta^T(t)\Pi(t)[A(t) - K(t)C(t)]\zeta(t) \\ & + 2\zeta^T(t)\Pi(t)n(t) \\ & + \mathrm{tr}\big([G(t)G^T(t) + K(t)D(t)D^T(t)]\Pi(t)\big) \end{aligned} \tag{78}$$

With eqns. 25, 30, 31 and 33 one can obtain

$$\|K(t)\| \le \|P(t)\| \, \|C(t)\| \, \|R^{-1}(t)\| \le \bar{k} \tag{79}$$

whereby $\bar{k} = \bar{p}\bar{c}/\underline{r}$. Moreover, from eqns. 37 and 38 one obtains

$$\mathrm{tr}[G(t)G^T(t)] \le \delta\mathrm{tr}[I] \le q\delta \tag{80}$$

$$\mathrm{tr}[D(t)D^T(t)] \le \delta\mathrm{tr}[I] \le m\delta \tag{81}$$

where q and m and the number of the rows of $G(t)$ and $D(t)$, respectively. Furthermore, using eqns. 31 and 79–81 it follows that

$$\begin{aligned} & \mathrm{tr}\big([G(t)G^T(t) + K(t)D(t)D^T(t)K^T(t)]\Pi(t)\big) \\ & \quad \le \frac{1}{\underline{p}}\mathrm{tr}[G(t)G^T(t)] + \frac{\bar{k}^2}{\underline{p}}\mathrm{tr}[D(t)D^T(t)] \\ & \quad \le k_{noise}\delta \end{aligned} \tag{82}$$

with

$$\kappa_{noise} = \frac{q}{\underline{p}} + \frac{\bar{k}^2 m}{\underline{p}} \tag{83}$$

Applying lemma 3.3 and using eqn. 25 leads to

$$\begin{aligned} \mathcal{L}V[\zeta(t), t)] \le {} & \zeta^T(t)\bigg[\frac{\partial\Pi}{\partial t}(t) + A^T(t)\Pi(t) + \Pi(t)A(t) \\ & - 2C^T(t)R^{-1}(t)C(t)\bigg]\zeta(t) \\ & + \kappa_{nonl}\|\zeta(t)\|^3 + \kappa_{noise}\delta \end{aligned} \tag{84}$$

for $\|\zeta(t)\| \le \varepsilon'$ with $\varepsilon' = \min(\epsilon_\phi, \epsilon_\chi)$. Calculate $d\Pi(t)$ by the formula

$$d\Pi(t) = -\Pi(t)dP(t)\Pi(t) \tag{85}$$

where eqn. 85 results from a differentiation of $\Pi(t)\,P(t) = I$ and $dP(t)$ is given by the Riccati differential equation (eqn. 12), and insert it into eqn. 84 This leads to

$$\begin{aligned} \mathcal{L}V[\zeta(t), t] \le {} & -\zeta^T(t)[\Pi(t)Q(t)\Pi(t) \\ & + C^T(t)R^{-1}(t)C(t)]\zeta(t) \\ & + \kappa_{nonl}\|\zeta(t)\|^3 + \kappa_{noise}\delta \end{aligned} \tag{86}$$

and, with eqns. 31 and 32 and $C^T(t)R^{-1}(t)C(t) \ge 0$, one obtains

$$\mathcal{L}V[\zeta(t), t]\} \le -\left(\frac{q}{\bar{p}^2} - \kappa_{nonl}\|\zeta(t)\|\right)\|\zeta(t)\|^2 + \kappa_{noise}\delta \tag{87}$$

Defining

$$\epsilon = \min\left(\epsilon', \frac{q}{2\kappa_{nonl}\bar{p}^2}\right) \tag{88}$$

and using eqn. 77 one obtains

$$\mathcal{L}V[\zeta(t), t)] \leq -\frac{qp}{2\bar{p}^2}V[\zeta(t), t)] + \kappa_{noise}\delta \tag{89}$$

for $\|\zeta(t)\| \leq \epsilon$. With eqns. 77 and 89 one can satisfy the requirements to apply lemma 2.3, where $\|\zeta(0)\| \leq \epsilon$, $\delta \leq \mu/\kappa_{noise}$, $\underline{v} = 1/\bar{p}$, $v = 1/\underline{p}$, and establish mean-square exponential boundedness as well as stochastic sample-path boundedness of the estimation error under the conditions of eqns. 36–38.