Bifurcation in Weighted Digraphs and Their Applications in Ecology

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BIFURCATION IN WEIGHTED DIGRAPHS AND THEIR APPLICATIONS IN ECOLOGY

by

Kehinde O. Irabor, M.S.

A Dissertation submitted to the Faculty of the Graduate School, Marquette University, in Partial Fulfillment of the Requirements for the degree of Doctor of Philosophy

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ABSTRACT
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Kehinde O. Irabor, M.S.
Marquette University, 2019

Merrill (2010) described bifurcation in a Markov chain by examining the eigenvalues of the associated probability matrix. The bifurcation point is that point where the dynamics of the system’s structure changes. He recognized a change in the dynamics of a sample path in a Markov chain when the nature of its eigenvalues changes. We built upon this work and found that not all changes in Markov chain dynamics are accompanied by change in the nature of the eigenvalues. And we introduce other measures that will recognize a change in dynamics. This was applied to solve the problem of evaluating the effectiveness of an ecological corridor. This was also used as a measure to examine bifurcation in metapopulation dynamics.

Ovaskeinan and Hanski (2003) gave four definitions of patch value (contribution of a patch to metapopulation dynamics and persistence). One of them denotes a patch value as $W_i$, the contribution of patch $i$ to colonization in the patch network. It is the left leading eigenvector of matrix $B$ whose entries, $b_{ij} = \frac{p_j c_{ij}}{\sum p_k c_{ik}}$. This is a Markov chain, where $p_i$ is the probability that patch $i$ is occupied, $c_{ij}$ is the contribution that occupied patch $j$ makes to the colonization rate of empty patch $i$. This matrix is in the family of coperiodic cospectral, which will be introduced in this dissertation. Therefore, it could be an effective tool in studying metapopulation dynamics.

The goal is to evaluate the effectiveness of corridor introduction on species persistence, richness, and ecosystem dynamics. We focused our application on available data from the Osceola-Ocala black bears in Florida.
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CHAPTER 1

INTRODUCTION TO GRAPHS, DIGRAPHS AND MATRICES

This Chapter provides a review of graphs, digraphs, and their matrix representations. The eigenvalues of their adjacency matrix representations are summarized, and this is also extended to Markov chains. Bifurcation in Markov chains is also discussed briefly. The contribution to this dissertation in this chapter are in Lemma 1.3.7, Theorem 1.3.8, Theorem 1.4.11, Theorem 1.4.14, and Theorem 1.4.15. Some of these contributions are not difficult, but they are original to the best of the Author’s knowledge.

1.1 Definitions

A graph consists of a non-empty set of points linked together by a set of line segments. The set of points is called a set of vertices (or nodes) and the line segments are called edges (see Figure 1(a)). A graph is used to model a binary relationship between a set of objects. The vertices represent the set of objects and the edges represent the binary relationship between the sets of objects. Therefore, there is an edge between a pair of vertices only if there is a binary relationship between them. The endpoints of an edge are the pair of vertices incident to it. For instance, the endpoints of edge $e_3$ in Figure 1(a) are vertices $v_4$ and $v_5$. The vertices joined to a vertex, $v$, by an edge are called the neighbors of $v$. That is, the neighbors of a vertex are adjacent to the vertex. For instance, the neighbors of vertex, $v_2$, in Figure 1(a) is the set $\{v_1, v_3, v_5\}$. They are adjacent to vertex $v_2$ (Gross & Yellen, 2006).
An edge that joins two different vertices is called a *proper edge*. An edge that joins a single endpoint to itself is a *self-loop or loop*. A *simple graph* is a graph with only proper edges and no self-loops. Two edges that have a vertex (an endpoint) in common are *adjacent edges*. The *degree* of a vertex $v$ can be calculated as the number of edges joined to it, in addition to twice the number of self-loops incident with it. A *complete graph* is a simple graph such that there is an edge between every pair of vertices (see Figure 1(b)). It is denoted by $K_n$, where $n$ is the number of vertices in the graph (Gross & Yellen, 2006).

A *directed graph* or *digraph* (see Figure 1(c)) is a graph with a direction on each of its edges. The directed edges are also called *arcs*. The arc models a *source-sink* relationship. The vertex adjacent to the arc end without the arrow is the *source* (or tail) of the relationship and the vertex adjacent to the arrow arc end is the *sink* (or head) of the relationship. Therefore, an arc is directed from its tail to its head. Ordered pair $(u, v)$ is used to denote an arc from vertex $u$ to vertex $v$. If two arcs between a pair of vertices have different directions, then they are *oppositely directed* (see Figure 1(c)) (Gross & Yellen, 2006).
The out-set of a vertex \( v \), is denoted as, \( O^+(v) \). It is the set of all vertices \( w \), such that \( (v,w) \) in an arc in the digraph (loops are also included). Similarly, the in-set of a vertex \( v \), denoted as, \( O^-(v) \), is the set of all vertices \( w \), such that \( (w,v) \) is an arc in the digraph. The indegree of a vertex \( v \), \( d^-(v) \), is the number of vertices in \( O^-(v), |O^-(v)| \) and the outdegree, \( d^+(v) \), is the number of vertices in \( O^+(v), |O^+(v)| \). In Figure 1(c), the in-set of vertex \( u \) is the set \{\( x, y, z \)\}. The outset of vertex \( u \) is the set, \{\( x, w \)\}. Therefore, the indegree of \( u \), \( d^-(u) \), is three and its outdegree, \( d^+(u) \), is two. A weighted digraph is a digraph such that each of its directed edges is assigned a number that represents the magnitude of the relationship modeled (Gross & Yellen, 2006). Most of the work in this dissertation will be on weighted digraphs.

### 1.1.1 Representation of Graphs and Digraphs as Matrices

Graphs and digraphs can be represented as matrices. The most common representations are incidence matrices, reachability matrices, and adjacency matrices. The incidence matrix, \( C = [c_{ij}] \), of a graph with \( v \) number of vertices and \( e \) number of edges is a \( v \times e \) matrix where the elements of the vertices represent the rows, and the element of the edges represent the columns of the matrix. Thus, entry, \( c_{ij} \), is 1 if the vertex represented by \( i \) is an endpoint of an edge represented by \( j \) and 0 otherwise. The
reachability matrix, \( R = [r_{ij}] \), of a digraph on \( n \) vertices is a \( n \times n \) matrix with entry, 
\( r_{ij} = 1 \) if there is path from vertex \( i \) to vertex \( j \), and 0 otherwise (Walter & Contreras, 1999, Gross & Yellen, 2006).

The adjacency matrix representation also has both its rows and columns representing the vertices. Thus, it is also a square matrix. It will be our focus in this dissertation. The adjacency matrix of a simple graph, \( A = a_{ij} \), on \( n \) vertices is an \( n \times n \) matrix that has 1 in both entry \( a_{ij} \) and entry \( a_{ji} \) if vertices, \( i \) and \( j \) are adjacent and 0 otherwise. Similarly, the adjacency matrix of a digraph on \( n \) vertices is an \( n \times n \) matrix in which entry \( a_{ij} = 1 \) if there is a directed edge, \( (i, j) \), from vertex \( i \) to vertex \( j \) and \( a_{ij} = 0 \) otherwise. For a weighted digraph, the entries will be the weight of the respective arcs. Walter and Contretras (1999) noted that any square matrix could be the adjacency matrix of a weighted digraph provided negative weights are allowed. A walk is an alternating sequence of vertices and edges in which each edge is adjacent to its endpoints in the sequence. The length of a walk is the number of edges in the walk sequence (Gross & Yellen, 2006).

**Powers of an adjacency matrix:** let \( A \) be the adjacency matrix of a digraph \( D \), then the \( i, j \) element of \( A^k \) is the number of walks of length \( k \) starting at vertex \( i \) and ending at vertex \( j \) in \( D \). For example, the adjacency matrix of the simple graph in Figure 1(a) and its third power is below.

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{bmatrix}
\]
In this example, entry $A^3(1,2) = 4$ represents the number of walks of length 3 from vertex $v_1$ to $v_2$. The four walks are:

$\langle v_1, e_2, v_5, e_2, v_1, e_1, v_2 \rangle$,
$\langle v_1, e_1, v_2, e_6, v_5, e_6, v_2 \rangle$,
$\langle v_1, e_1, v_2, e_5, v_3, e_5, v_2 \rangle$, and
$\langle v_1, e_1, v_2, e_1, v_1, e_1, v_2 \rangle$.

Two vertices $u$ and $v$ in a digraph $D$ are mutually reachable if there is a directed path from $u$ to $v$ and a directed path from $v$ to $u$. A digraph is strongly connected if all its vertices are mutually reachable. A strong component of a digraph $D$ is a "subdigraph of $D$ induced on a maximal set of mutually reachable vertices" (Gross & Yellen, 2006).

The adjacency matrix of a digraph with more than one strong component can always be permuted to look like any of the following matrix equations:

$$ A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (1.1) $$

Where $A_{11}$ is the adjacency matrix of a strong component of $D$, labeled $C_1$. In equation (1.1), vertices in $C_1$ may reach vertices in $D - C_1$ (whose adjacency matrix is $A_{22}$) but not vice versa. Another permutation of adjacency matrix of $D$ is

$$ A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad (1.2) $$

Similarly, in equation (1.2), vertices in $D - C_1$ may reach $C_1$, but not vice versa.

$$ A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad (1.3) $$
Finally, vertices in $C_1$ and $D - C_1$ may not reach each other. In this case, the digraph is disconnected. Observe that each of $A_{11}$ and $A_{22}$ could represent more than one component. For instance, an adjacency matrix, $A$, with 3 components may have the following form:

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix}$$  \hspace{1cm} (1.4)

Walter and Contreras (1999) also showed and proved some relationships between powers of an adjacency matrix and its reachability matrix. The relationship between powers of a reachability matrix and its strong component interpretation was also discussed (see Walter and Contreras, 1999, P. 54-56).

A directed cycle of length $n$ is a digraph with $n$ vertices and $n$ directed arcs such that each vertex has an in-degree and out-degree of 1 (each vertex has only one source and one sink). For instance, the digraph in the figure below, is a directed cycle of length 4. It has four vertices and four edges. Also, the in-degree and out-degree of each vertex is one.

**Figure 1d(i):** A directed cycle of length 4

![Figure 1d(i): A directed cycle of length 4](image-url)
1.2 Eigenvalues and Eigenvectors

The eigenvalues of an \( n \times n \) matrix, \( A \), are well known as scalars, \( \lambda_1, \lambda_2, \ldots, \lambda_n \), that satisfy:

\[
Ax = \lambda x
\]  

(1.5)

Where \( x \) is a nonzero column vector known as an eigenvector. More specifically, this is a right eigenvector because left eigenvectors also exist as row vectors, \( x_L \), such that

\[
x_L A = \lambda x_L.
\]  

(1.6)

The eigenvalues are the roots of the characteristic polynomial of matrix \( A \) and satisfy \( \det(A - \lambda I) = 0 \), the characteristic equation. Thus, an \( n \times n \) matrix will have \( n \) eigenvalues, counted with multiplicity. This is an application of the Fundamental Theorem of Algebra. The eigenvalues could be all distinct or some repeated. Eigenvalues could also be real numbers or complex numbers. The set of the eigenvalues of a matrix is known as its spectrum. Some well-known theorems of matrix eigenvalues are listed below.

**Theorem 1.2.1** (Gantmacher, 1959, Cayley, 1889, Sylvester, 1883). Let \( A \) be an \( n \times n \) matrix and \( \lambda \) be its eigenvalues. Then, the following statements are true:

(i) The sum of the \( n \) eigenvalues of \( A \) is the same as sum of the diagonal elements of \( A \) (the trace of \( A \)).

(ii) The product of the \( n \) eigenvalues of \( A \) is the determinant of \( A \).

(iii) If \( a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \) is the characteristic polynomial of \( A \), then

\[
a_{n-1} = -\text{trace}(A) \quad \text{and} \quad a_0 = (-1)^n\det(A).
\]
(iv) **Sylvester’s Matrix Theorem** states that $f(A) = \sum_{i=1}^{k} f(\lambda_i)A_i$. Where, $f(A)$ is a function of matrix $A$, $\lambda_i$ is the eigenvalues and $A_i$ is the corresponding Frobenious covariant of $A$.

(v) **The Cayley-Hamilton Theorem** states that if $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ is the characteristic polynomial of $A$, then the matrix equation

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0.$$  

The relationship between eigenvalues and strong components can be seen in Theorem 1.2.2 below.

**Theorem 1.2.2** (Walter and Contretras, 1999). *Let $A$ be the adjacency matrix of a digraph, $D$. Then, the number $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of the adjacency matrix of a strong component of $D$.*

**Proof**

Let $C_1$ be a strong component of digraph $D$ and $A_{11}$ be its adjacency matrix. Hence, a permutation of the adjacency matrix of $D$ can be represented as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$  

Since, this is an “if and only if” theorem, there are two parts to the proof.

(i) Suppose, $\lambda$ is an eigenvalue of $A_{11}$. Then, the characteristic equation of $A$ is the determinant of

$$A - \lambda I = \begin{bmatrix} A_{11} - \lambda I_{11} & A_{12} \\ 0 & A_{22} - \lambda I_{22} \end{bmatrix}.$$  

Where $I_{11}$ and $I_{22}$ are appropriately sized identity matrices. Also, $0$ and $A_{12}$ are matrices.

Then, $\det(A - \lambda I) = \det(A_{11} - \lambda I_{11}) \det(A_{22} - \lambda I_{22})$
\[ = \det(A_{11} - \lambda I_{11}) \det(A_{22} - \lambda I_{22}) \]

Thus, \( \det(A_{11} - \lambda I_{11}) = 0 \) or \( \det(A_{22} - \lambda I_{22}) = 0 \). \hspace{1cm} (1.8)

Hence, \( \{ \text{eigenvalues of } A \} = \{ \text{eigenvalues of } A_{11} \} \cup \{ \text{eigenvalues of } A_{22} \} \). Thus, if \( \lambda \) is an eigenvalue of \( A_{11} \) (or \( A_{22} \)), it is an eigenvalue of \( A \).

Therefore, the eigenvalues of the adjacency matrix of a strong component, \( C_1 \), are also the eigenvalues of the adjacency matrix of \( D \).

(ii) To prove that \( \lambda \) is an eigenvalue of a strong component \( C_1 \) if it is an eigenvalue of \( D \), we have two cases.

Case 1: Let \( D \) be a digraph with only one strong component. This implies that \( C_1 = D \). Thus, the eigenvalues of \( D \) are also the eigenvalues of its strong component, \( C_1 \).

Case 2: Suppose \( D \) has more than one strong component and let \( C_1 \) be one of the strong components. Then, a permutation of the adjacency matrix of \( D \) is of the form:

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
\]

Where \( A_{11} \) represents the adjacency matrix of a strong component, \( C_1 \) and \( A_{22} \) represents the adjacency matrix of the remaining strong component or components, \( (D - C_1) \).

Hence, the characteristic polynomial of \( A \) is:

\[
\det(A_{11} - \lambda I_{11}) \det(A_{22} - \lambda I_{22}) = 0 \text{ from } (1.8)
\]

That is, \( \det(A_{11} - \lambda I_{11}) = 0 \) or \( \det(A_{22} - \lambda I_{22}) = 0 \)

Hence, \( \lambda \) is eigenvalue of \( A_{11} \) or \( A_{22} \).

Thus, \( \lambda \) is an eigenvalue of a strong component of \( D \) if it is an eigenvalue of \( A \).
**Corollary 1.2.3** (Walter and Contretras, 1999). *The eigenvalues of the adjacency matrix of a digraph, \(D\), are the eigenvalues of the adjacency matrix of all its strong components.*

**Proof**

The steps followed in the proof of Theorem 1.2.2 can be repeated for \((D - C_1)\) represented by matrix \(A_{22}\) to find that the characteristic polynomial of \(A\) factors as:

\[
\det(A_{11} - \lambda I_{11}) \det(A_{22} - \lambda I_{22}) \ldots \det(A_s - \lambda I_s) = 0 \tag{1.9}
\]

### 1.2.1 Transpose Similarity

A matrix, \(A\), is similar to another matrix \(B\) if there exist an invertible matrix, \(P\), such that

\[
P^{-1}AP = B. \tag{1.10}
\]

That is, \(AP = PB\). The transformation of \(A\) into \(P^{-1}AP\) is called similarity transformation.

**Theorem 1.2.4** (Lay, Lay & McDonald, 2016). *If \(n \times n\) matrices \(A\) and \(B\) are similar, then they have the same characteristic polynomial and hence the same eigenvalues.*

**Proof**

A proof of this theorem can be found in Lay, Lay and McDonald (2016).

A matrix and its transpose are similar. Hence, the eigenvalues of a matrix, \(A\), is the eigenvalues of its transpose.

### 1.3 Spectra Characterization of Graphs

Graphs with the same adjacency matrix spectra are called *isospectral graphs* (or cospectral). Some graphs can be characterized by their spectrum up to an isomorphism. For instance, any regular graph of degree 2 and the complete multipartite graph \(K_{n,n,...,n}\)
(for any positive integer \( n \)) are characterized by their spectrum up to an isomorphism (Cvetkovic, et al. 2010). Some family of graphs can also be determined by their spectrum. For instance, the family of the line graph of projective plane and affine plane of order \( n \) are cospectral (Hoffman, 1965 and Hoffman & Ray-Chaudhuri, 1965).

Some graph properties are not uniquely determined by the graph’s spectrum. Literature has used the terms PINGs (Pair of Isospectral Non-isomorphic Graphs) and SINGs (Set of Isospectral Non-isomorphic Graphs) to describe these graphs. Although the spectrum does not determine a graph uniquely, it gives some structural information about the graphs beyond the degree. For instance, consider a regular graph, \( G \), of degree 2. The spectrum gives more structural information than the degree because it determines the length of all cycles in \( G \) (like the second largest eigenvalue of the adjacency matrix determines the length of the largest cycle in \( G \)). This eventually determines \( G \) up to an isomorphism (Cvetkovic, et al. 2010).

Spectral characterization can also be extended to digraphs. Walter and Contretras (1999) show that the characteristic equation of any directed cycle of length \( n \) satisfies

\[
\lambda^n = 1.
\]

Proposition 1.3.5 (Walter and Contretras, 1999). The eigenvalues of a directed cycle on \( n \) vertices are the \( n \)th roots of unity.

Proof

The characteristic equation of the adjacency matrix of a directed cycle on \( n \) vertices satisfies, \( \lambda^n - 1 = 0 \), so \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are \( n \)th roots of unity. \( \blacksquare \)
**Example 1.3.6** The eigenvalues of the directed cycle on 3 vertices below is the solution of its characteristic polynomial, \( \lambda^3 - 1 = 0 \). This is the same as \( \lambda^3 = 1 \), solutions are the cube roots of unity. Thus, \( \lambda = 1, -0.5 + i\sqrt{0.75}, \) and \( -0.5 - i\sqrt{0.75} \).

**Figure 1d:** A directed cycle on 3 vertices

This dissertation extended this result to weighted cycles and weighted digraphs with simple cycles as a strong component.

**Lemma 1.3.7** *The eigenvalues of any weighted directed cycle of length n must satisfy*

\[
\lambda^n = \omega
\]  

*(1.11)*

*where \( \omega \) is the product of the weights of the cycle.*

**Proof**

Let \( a_1, a_2, a_3, ..., a_n \) represent the weights of the directed edges on a cycle. Then, \( \omega = a_1 a_2 a_3 ... a_n \) (the product of the directed edge weights). Since \( A \) is the adjacency matrix of a weighted cycle digraph, then \( A - \lambda I \) is an \( n \times n \) upper bidiagonal matrix with an additional non zero entry in \((n, 1)\). The diagonal entries are \(-\lambda\) and the upper diagonal entries are the edge weights such that entry \((n - 1, n)\) is the edge weight from vertex \( n - 1 \) to vertex \( n \). The \((n, 1)\) entry, \( a_n \), is the edge weight from vertex \( n \) to vertex 1.
Then, we compute the determinant of $A - \lambda I$ along the first column. The first column has only 2 non-zero entries, the first and last row.

The determinant of $A - \lambda I$ along the first column will be:

$$
\det(A - \lambda I) =
$$

\[
\begin{pmatrix}
-\lambda & a_1 & 0 & \ldots & 0 & 0 \\
0 & -\lambda & a_2 & \ldots & 0 & 0 \\
0 & 0 & -\lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -\lambda & a_{n-1} \\
a_n & 0 & 0 & \ldots & 0 & -\lambda
\end{pmatrix}
\]

**Case 1:** If $n$ is even, $(-1)^{n+1} = -1$

Since a bidiagonal matrix is a triangular matrix, its determinant is the product of its diagonals. Hence,

$$
\det(A - \lambda I) = (-\lambda(-\lambda)^{n-1}) - a_n(a_1a_2 \ldots a_{n-1})
$$

$$
= \lambda^n - a_1a_2a_3 \ldots a_n.
$$

Thus, the characteristic equation is:

$$
\lambda^n - a_1a_2a_3 \ldots a_n = 0.
$$

Hence, $\lambda^n = a_1a_2a_3 \ldots a_n$.

Therefore, $\lambda^n = \omega$ is satisfied.
Case 2: if $n$ is odd.

$$(-1)^{n+1} = 1$$

The determinant of $A - \lambda I$ along the first column will be

$$\det(A - \lambda I) = (-\lambda (-\lambda)^{n-1}) + a_n (a_1 a_2 \ldots a_{n-1})$$

$$= -\lambda^n + a_1 a_2 a_3 \ldots a_n.$$  

Thus, the characteristic equation is

$$-\lambda^n + a_1 a_2 a_3 \ldots a_n = 0.$$  

(1.12)

Hence, $\lambda^n = a_1 a_2 a_3 \ldots a_n$

Therefore, the eigenvalues of a weighted cycle digraph must satisfy $\lambda^n = \omega$. Where $\omega$ is the product of the edge weights.

**Theorem 1.3.8** Let $A$ be the adjacency matrix of a weighted digraph, $D$. Then, the characteristic polynomial of matrix $A$ has a factor $(\lambda^n - \omega)$ if and only if there is a strong component of $D$ that is a weighted directed cycle of length $n$ and the product of its weights is $\omega$.

**Proof**

(i) Suppose a weighted directed cycle of length $n$ is a strong component of digraph, $D$. Then, the eigenvalues of its adjacency matrix satisfy $\lambda^n = \omega$ by Lemma 1.3.7. Hence, we have $\lambda^n - \omega = 0$. The eigenvalues of $D$ will also satisfy $\lambda^n = \omega$ by Theorem 1.2.2. Thus, $\lambda^n - \omega$ will be a factor of the characteristic polynomial of $D$.

(ii) Suppose the characteristic polynomial of $D$ factors as $(\lambda^n - \omega)$. Then, some of the eigenvalues of $D$ satisfy $\lambda^n = \omega$. Hence, there is a strong component of $D$
whose eigenvalues satisfy $\lambda^n = \omega$. This strong component must be a weighted directed cycle by Lemma 1.3.7.

1.4 Markov Chains and Probability Matrices

A Markov chain (or stochastic digraph) was introduced by Markov in 1906. It is a special type of weighted digraph such that the weight of each edge is nonnegative and the total out-degree weight of each vertex is one. The vertices of a Markov chain are usually known as states and represented by state 0 ($s_0$) to state $n - 1$ ($s_{n-1}$) for any n-state Markov chain. Each directed arc weight $w_{(i,j)}$ represents the probability of moving from the source state, $s_i$, to the sink state, $s_j$ in one step. The adjacency matrix of a Markov chain is known as its probability matrix (or transition matrix). Thus, each entry of the probability matrix is nonnegative, less than or equal to 1, and each row sums to 1 (Markov, 1906). For instance, the probability matrix of the Markov chain in Figure 1(e) is

$$
\begin{bmatrix}
\frac{3}{5} & \frac{2}{5} \\
\frac{1}{3} & \frac{2}{3}
\end{bmatrix}
$$

The probability matrix can also be used to generate a state-to-state sample paths over time (Merrill, 2010). These are the set of outcomes generated by running a simulation of a probability matrix. See more on sample path approach in Chapter 4.4.

![Figure 1e: A Markov chain](image)
**Theorem 1.4.9** (Markov, 1906, Merrill, 1998, Plavnick, 2008). \textit{The Fundamental Theorem of Markov Chains} states that if $P$ is the probability matrix of a Markov chain, then $\lambda_1 = 1$ is always an eigenvalue, the modulus of all the eigenvalues are less than or equal to one and a limit distribution exists if $\lambda_1 = 1$ is the only eigenvalue with modulus 1.

**Proof**

The Standard result and proof of this theorem can be found in Markov, 1906, Merrill, 1998, and Plavnick, 2008.

Let $P$ be the probability matrix of a Markov chain, then the $(i, j)$ element of $P^k$ (kth power of the transition matrix) is the probability of moving from state $i$ to state $j$ in $k$ steps. As $k$ tends to infinity, each row of $P^k$ tends to the \textit{limit distribution}, if it exists. The limit distribution is a vector in which each of its components represents the fraction of time a sample spends at each state (vertex) of the Markov chain. That is, each component, $x_i$ of the limit vector, $x$ represents:

$$x_i = \lim_{n \to \infty} \left( \frac{v_i}{n} \right)$$  \hfill (1.13)

Where $v_i$ is the number of visits to state $i$ in the first $n$ steps. The limit distribution, $x$ satisfies the left-handed system of equation,

$$xP = x$$  \hfill (1.14)

And $x$ is a row probability vector. This can be transformed to the right-handed system as:

$$P'x' = x'$$  \hfill (1.15)

where $x'$ is now a column vector.

Hence, $x$ is a left eigenvector of eigenvalue, $\lambda = 1$.

Below, we present the eigenvalues and limit distribution (when it exists) for a 2-state Markov chain, cycle Markov chain and a 3-state Markov chain with no self-loops.
1.4.1 Eigenvalues and Limit Distribution of Some Markov Chain

This section observes the eigenvalues of some Markov chains, its limit distribution and relationship.

**Proposition 1.4.10** Let M be a 2-state Markov chain, then the eigenvalues of M are:

\[
\lambda_1 = 1 \text{ and } \\
\lambda_2 = \sum p_{ii} - 1.
\]

Where, \( p_{ii} \) is the probability of self-loops.

**Proof**

Let the probability matrix of M be:

\[
P = \begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}
\]

The sum of the self-loop probabilities, \( p_{11} + p_{22} \), is the trace of the probability matrix. By Theorem 1.4.9, the maximum eigenvalue of M is \( \lambda_1 = 1 \). Also, by Theorem 1.2.1(i),

\[
\lambda_1 + \lambda_2 = p_{11} + p_{22}
\]

Therefore, \( \lambda_2 = p_{11} + p_{22} - \lambda_1 \)

\[
= p_{11} + p_{22} - 1 \quad (\text{since } \lambda_1 = 1)
\]

\[
= \sum p_{ii} - 1. \quad \blacksquare
\]

**Theorem 1.4.11** Let \( P \) be the probability matrix of a two-state Markov chain, \( \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \),

Then, the vector \( \begin{bmatrix} 1 \\ p_{12} \\ p_{21} \end{bmatrix} \) and/or \( \begin{bmatrix} p_{21} \\ p_{12} \\ 1 \end{bmatrix} \) when normalized (by dividing each element by its sum)

will satisfy the limit distribution if \( \lambda_1 = 1 \) is the only eigenvalue of modulus \( = 1 \).

(Note: This theorem is also established by Sylvester’s matrix theorem in Theorem 1.2.1(iv)).
Proof

\[ P' = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \quad (1.16) \]

Let \( x' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) be the limit distribution.

Then \( P'x = x \) (same as \( xP = x \) when \( x \) is a row vector), which represents the following simultaneous equations.

\[ p_{11}x_1 + p_{21}x_2 = x_1 \quad (1.17) \]
\[ p_{12}x_1 + p_{22}x_2 = x_2 \quad (1.18) \]

Solving equation (1.17) yields \( p_{21}x_2 = (1 - p_{11})x_1 \), but as \( 1 - p_{11} = p_{12} \) since \( p_{11} + p_{12} = 1 \) (row stochastic).

\[ p_{21}x_2 = p_{12}x_1 \]

Hence,

\[ x_2 = \frac{p_{12}x_1}{p_{21}} \]

And therefore, \( x' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{p_{12}x_1}{p_{21}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{p_{12}}{p_{21}} \end{bmatrix} x_1 \)

Similarly, \( \begin{bmatrix} p_{21} \\ p_{12} \\ 1 \end{bmatrix} \) can be derived by solving \( p_{12}x_1 + p_{22}x_2 = x_2 \) for \( x_2 \).

Example 1.4.12: The eigenvalues of the Markov chain in Figure 1(e) are:

\[ \lambda_1 = 1 \text{ and } \lambda_2 = \left( \frac{3}{5} + \frac{2}{3} \right) - 1 = \frac{4}{15}. \]

Since \( \frac{p_{12}}{p_{21}} = \frac{2/5}{1/3} = \frac{6}{5} \), when \( \begin{bmatrix} 1 \\ \frac{6}{5} \\ \frac{11}{5} \end{bmatrix} \) is normalized, it satisfies the limit distribution, \( \begin{bmatrix} 5 \\ 11 \\ 6 \\ 11 \end{bmatrix} \).
Similarly, the normalized form of \[
\begin{bmatrix}
\frac{5}{6} \\
1
\end{bmatrix}
\] from
\[
\begin{bmatrix}
p_{21} \\
p_{12}
\end{bmatrix}
\]
(since \(\frac{p_{21}}{p_{12}} = \frac{1/3}{2/5} = \frac{5}{6}\)) is
\[
\begin{bmatrix}
\frac{5}{11} \\
\frac{6}{11}
\end{bmatrix}.
\]

**Example 1.4.13** The Markov chain below has 2 strong components, a cycle of length 3 and a cycle of length 2. By the theorems above, we can tell that the 5 eigenvalues of the weighted adjacency matrix satisfy:

\[
\lambda^2 = 1 \quad \text{and} \quad \lambda^3 = 0.7.
\]

Hence, the eigenvalues are \(1, -1, \sqrt{0.7}, \text{and approximately } -0.44395 \pm 0.768947i\). The first 2 eigenvalues satisfy \(\lambda^2 = 1\) and the last 3 satisfy \(\lambda^3 = 0.7\).

**Figure 1f**: Markov chain with 2 strong components

**Theorem 1.4.14** Suppose \(M\) is a 3-state Markov chain and no self-loops, then the eigenvalues of the probability matrix \(P\) must satisfy:

\[
\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 + \lambda_3 &= -1 \\
\lambda_2 \ast \lambda_3 &= k
\end{align*}
\]

where \(k\) is the sum of the weight product of the two cycles of length 3.

**Proof**

By Theorem 1.4.9, \(\lambda_1 = 1\), which proves equation (1.19). By Theorem 1.2.1(i),

\[
\lambda_1 + \lambda_2 + \lambda_3 = \text{trace} (P).
\]

Since, \(P\) has no self-loops, \(\text{trace}(P) = 0\). Thus,

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0
\]

(1.22)
By substituting (1.19) in (1.22), we have, \( \lambda_2 + \lambda_3 = -1 \) which proves equation (1.20).

By Theorem 1.2.1(ii), \( \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \), is the determinant of matrix P. Since, \( \lambda_1 = 1 \), we have
\[
\lambda_2 \cdot \lambda_3 = \det(P)
\]

Observe that since \( M \) is a 3-state chain with no self-loops, the determinant is the sum of the product of the two cycle weights, which is represented by \( k \). Thus,
\[
\lambda_2 \cdot \lambda_3 = k
\]

**Theorem 1.4.15** Let \( M \) be a three-state Markov chain with no self loops. If \( \lambda_1 = 1 \) is the only eigenvalue of modulus 1, then the vector, \[
\begin{bmatrix}
1 - v_{-0} \\
1 - v_{-1} \\
1 - v_{-2}
\end{bmatrix}
\]
is an eigenvector and when it is normalized, it is the limiting distribution. Where \( v_{-i} \) is the product of nonadjacent edges to state \( i \).

![Figure 1g: Stochastic digraph with no self-loop](image)

**Proof**

Let \( P = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \) be the probability matrix of \( M \) in Figure 1(g). And let \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) be the limiting distribution of \( M \). Then, \( xP = x \). This can also be written as:
\[
P^T x' = x'
\] (1.23)
\[
\begin{bmatrix}
0 & c & e \\
a & 0 & f \\
b & d & 0
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix}
\]

\[cx_2 + ex_3 = x_1\]  \hspace{1cm} (1.24)

\[ax_1 + fx_3 = x_2\]  \hspace{1cm} (1.25)

\[bx_1 + dx_2 = x_3\]  \hspace{1cm} (1.26)

By substituting equation (1.26) into (1.25) and solving for \(x_2\),

\[x_2 = \left(\frac{a + bf}{1 - df}\right)x_1.\]

Similarly, \(x_3 = \left(\frac{b + ad}{1 - df}\right)x_1\).

Therefore,

\[
\begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix} = \begin{bmatrix} 1 - df \\
an + bf \\
b + ad \end{bmatrix} \left(\frac{1}{1 - df}\right)x_1.
\]

Hence, an eigenvector satisfies

\[
\begin{bmatrix} 1 - df \\
an + bf \\
b + ad \end{bmatrix}.
\]

Note that

\[a + bf = a + b(1 - e) = a + b - be = 1 - be.\]

Similarly,

\[b + ad = b + a(1 - c) = b + a - ac = 1 - ac.\]

Therefore, an eigenvector satisfies
Observe that \( v_{-0} = df \) is the product of nonadjacent edges to state, \( s_0 \). Similarly, for \( v_{-1} = be \) and \( v_{-2} = ac \).

This becomes a limiting distribution when each element is divided by the column sum.

**Example 1.4.16:** The adjacency matrix of the Markov chain below has eigenvalues:

\[
\lambda_1 = 1, \lambda_2 = -0.5 + \sqrt{10^{-3}}, \lambda_3 = -0.5 - \sqrt{10^{-3}}
\]

\[
\begin{bmatrix}
0 & 0.44 & 0.56 \\
0.7 & 0 & 0.3 \\
0.55 & 0.45 & 0
\end{bmatrix}
\]

These eigenvalues satisfy the equations in Theorem 1.4.14 above because

\[
\lambda_2 + \lambda_3 = -1 \quad \text{and} \quad \lambda_2 \ast \lambda_3 = 0.249.
\]

Note that, \((0.44 \ast 0.3 \ast 0.55) + (0.56 \ast 0.45 \ast 0.7) = 0.249\).

Also, \[
\begin{bmatrix}
1 - 0.45 \ast 0.3 \\
1 - 0.55 \ast 0.56 \\
1 - 0.44 \ast 0.7
\end{bmatrix} = \begin{bmatrix}
0.865 \\
0.692 \\
0.692
\end{bmatrix}
\]

is an eigenvector. When it is normalized, it becomes \[
\begin{bmatrix}
\frac{865}{2249} \\
\frac{692}{2249} \\
\frac{692}{2249}
\end{bmatrix}
\]

(approximately, \[
\begin{bmatrix}
0.385 \\
0.308 \\
0.308
\end{bmatrix}
\]
), which is the limit distribution.
The vertices in a strong component of a Markov chain form a *communicating class*. The communicating class is closed if none of its vertex set can reach a vertex in another communicating class; otherwise, it is open (Wolff, 1989).

Observe that an irreducible cycle Markov chain has no limit distribution because all (more than one) of its eigenvalues have modulus 1, since they satisfy $\lambda^n = 1$. All roots of unity have modulus 1.

**1.4.2 Powers of the Matrices of Markov chains and their Spectra**

The *n-step transition probability*, $p_{ij}^{(n)}$, is the probability of moving from state $i$ to state $j$ in $n$ time steps. That is, $p_{ij}^{(n)} = \text{prob}\{X_n = j|X_0 = i\}$ (Allen, 2003). Allen (2003) also showed that the $n$th power of a transition matrix, $P^n$, is also the *$n$-step transition matrix*, $P^{(n)} = (p_{ij}^{(n)})$. That is $P^{(n)} = P^n$. The Chapman-Kolmogorov equation was also used to show that $P^{(n)} = P^{(n-s)}P^{(s)}$.

Since the powers of the matrices of a Markov chain, $P^n$, are the same as the $n$-step probabilities, $P^{(n)}$, the eigenvalues are also the same.

Thus, if $\lambda$ is an eigenvalue of $P$, $\lambda^m$ is an eigenvalue of $P^m$. In addition, the limit distribution remains the same.

**1.5 Bifurcation in a Markov Chain**

Merrill (2010) defined bifurcation in a Markov chain as a “change in the dynamics of an underlying system with a changing parameter”. There is a change in the topology of the digraph. Systems with similar dynamics will have a similar blend of complex, negative and positive eigenvalues. The limit distribution will have the same
number of nonzero values in equivalent states, a similar autocorrelation pattern and small
differences in spectral radius (Merrill, 2010). Therefore, there can be different types of
bifurcation as a parameter in a chain change.

One context is the change in the nature of the eigenvalues of the probability
matrix. It can indicate the length of the dominant cycle path in the chain. For instance, a
negative eigenvalue will indicate a cycle path of length 2 periodicity, complex
eigenvalues with negative real part indicate a cycle path of length 3 periodicity, and
purely imaginary eigenvalues indicate a cycle path of length 4 periodicity (Merrill, 2010).
By proposition 1.3.5, the eigenvalues of a directed cycle on n vertices are the nth root of
unity. A directed cycle on two vertices is a loop 2 periodicity. Its eigenvalues will satisfy
\[ \lambda^2 = 1. \] Thus, \( \lambda = 1, -1 \), a negative eigenvalue. Similarly, a cycle on 3 vertices is a
period 3 loop, and its eigenvalues satisfy \( \lambda^3 = 1 \), which are complex eigenvalues with a
negative real part. Lastly, a cycle on 4 vertices is a period 4 loop, and its eigenvalues
satisfy \( \lambda^4 = 1 \), which are \( \lambda = 1, -1, i, -i \).

Bifurcation can be detected by the occurrence or disappearance of a complex
eigenvalue (change in approach to limit distribution), appearance of a root of unity
(apparance of new loop), and the likes (Merrill, 2010). These concepts will be discussed
further in Chapters 3 and 4.

The goal of the dissertation is to expand upon the work of Merrill (2010) by
giving mathematical proofs and extending the concept to other digraph topologies. The
dissertation completely characterizes the eigenvalues of 3-state Markov chains and shows
how digraph topology or loop structure determines or relates with eigenvalue nature.
These and other extensions will be applied to solve landscape connectivity problems in ecology.

A population of populations separated spatially but connected by movement of individuals is known as a metapopulation (Hanski, 1999). The focused application in this dissertation is metapopulation and corridor ecology. A digraph is used to model the relationship between habitat patches. The vertices represent the habitat patches, and the weighted arcs represent the rate or probability of movement between the patches. Therefore, we attempt to use graph theory, the digraph’s adjacency matrix, and its extension to model corridor connections between habitat patches to form an optimal metapopulation.
Chapter 2

INTRODUCTION AND DISCUSSION OF ECOLOGY APPLICATIONS

This Chapter introduces the Ecology application of this dissertation. It also gives a literature review of the works that have been done on Metapopulation, corridors and connectivity of fragmented landscapes.

2.1 Basic Definitions

An *ecosystem* is the interaction between living organisms and the non-living components in their environment. *Habitat* is the specific environment and condition that is befitting or acceptable by a particular species. It is also defined broadly as the amount of native vegetation cover (Lindenmayer & Hobbs, 2007).

*Habitat fragmentation* is usually a result of human activities such as roads cutting through a forest, housing development etc. Fragmentation reduces the available area for species and divides an entire habitat into smaller fragmented patches. Each species has a required amount of habitat size for it to persist, which is often determined by Population Viability Analysis (PVA). However, Fahrin (in Lindenmayer & Hobbs, 2007) noted that this method is not sufficient because it ignores the effect of emigration and mortality.

*Metapopulation* is a group of local population living in spatially isolated patches and connected by exchange of individuals among patches (Hanski, 1999). *Habitat corridor* is used to reduce the negative effect of habitat fragmentation by making it a metapopulation. It involves connecting fragments or patches of habitat by corridors of similar habitat to enhance exchange of individuals (Molles, 2008).
2.2 Problem Discussion: Connectivity and Corridor

Habitat loss and fragmentation has been identified as one of the major threats to the survival of species. Therefore, there is a strong interest in reducing the effect of fragmentation and tools that can predict the effect of management projects and infrastructural work on connectivity (Adriaensen et al, 2003). Fragmentation reduces the area available to species and the species are unable to move from one fragmented part to the other. This reduces the survival probability of most species.

It has been discovered that the survival of species in fragmented habitats depends on the interchanging of species between the fragments (recolonization rate). One approach of facilitating exchange is the use of habitat corridors to connect fragmented habitat (Tischendorf & Wissel, 1997; Adriaensen et al. 2003; Beier et al. 2008). However, there has been some controversy over the effectiveness of corridor in maintaining connectivity and biodiversity (Tischendorf & Wissel, 1997; Falcy and Estades, 2007)

Falcy and Estades (2007) conducted a simulation to check the effectiveness of corridors relative to habitat patch enlargement. They identified and used four factors for the simulation. The factors are, distance or patch isolation, patch size, corridor width and the probability of habitat-matrix crossing The population dynamics were modelled by using Energy-balance approach (Jekeli, 2017), which is governed by availability of food. Each individual expend energy at a constant rate. When the energy falls below a threshold, the individual dies and is removed from the simulation. When an individual reaches a cell with food, its energy increases. When the energy is above a particular
threshold, the individual reproduces at any contact with an opposite-gendered individual of equally above threshold energy level (Falcy and Estades 2007).

The authors compared habitat extension and corridor effectiveness by using a corridor to connect a fragmented patch to a larger habitat source. The population size of a similar habitat patch that was not connected, but extended on the edges by same area as the corridor was then compared with the population of the corridor-connected-patch. The authors concluded that if a patch size is close to the minimum required for population viability, patch enlargement is more effective. However, if the patch size is very small relative to the minimum required population viability size, corridors are more effective. The corridor increased the mean population size by population supplementation. Its width does not have any effect on the relative benefit. Therefore, their work is helpful in determining the right conservation strategy to use when resources are limited (Falcy and Estades 2007).

Tischendorf & Wissel (1997) used individual movement simulation to hypothetically check the functionality of a corridor as a channel of movement for small mammals. Empirical data from tracking was used to determine the movement parameters of the simulation. Species’ attainable distance through a corridor was determined by movement pattern (using step length and step angle as parameters), boundary returning angle (reflection, turn around or turn back), and corridor width. The possibility of a moving individual reaching a target distance area within a given time was defined as the transition probability. It was concluded that the transition probability depends on degree of movement autocorrelation (angle between two consecutive movement steps). As movement autocorrelation increases, boundary returning angle and corridor width begin
to influence the transition probability. Corridor width was varied to observe how it affect transition probability. This was used to determine an optimum corridor by focusing on corridors movement canalization, en route mortality and movement velocity (Tischendorf & Wissel 1997).

The corridor width determines the frequency of boundary encounters. It was called the relative boundary pressure (BP). It was calculated as the mean frequency of boundary encounter per movement per time step. It was found to be linearly correlated with perimeter: area ratio (P/A). Therefore, the corridor will be the easiest way to modify the transition probability. Although boundary encounter slows down movement, it was determined that the canalization of movement by corridor further increases the transition probability. This was determined by comparing the geometric ratio between a target area and circle ring area with transition probability of corridors with different movement pattern (Tischendorf & Wissel 1997).

Dixon, et al. (2006) used noninvasive sampling and genetic approach to evaluate the effectiveness of Osceola-Ocala regional corridor on Florida black bear population. Hair snares, with baits, were placed on different GIS map cells of the corridor. The hair snares take bear’s hair with follicles, which were genetically analyzed. The authors used population-assignment tests to assign individuals to either Osceola or Ocala origin. It was concluded that the corridor was effective in only one direction. This is because Osceola bears were not found in Ocala, but Ocala bears were found in Osceola. Also, some bears in Osceola were genetically from Osceola-Ocala mating. The unidirectional bear movement could be as a result of the highway on the corridor, which is closer to the Osceola habitat (Dixon, et al., 2006).
More recently, Green et al. (2018) evaluated the effectiveness of a Kenyan wildlife corridor as either a corridor link or a habitat extension. The authors used data on elephant behavior on the Mount Kenya Elephant Corridor. Camera traps were placed at about 500 m² grid on identified elephant trails on the corridor. The cameras recorded photos when triggered by movement. Elephant’s travel speed between two camera sites was estimated by dividing the distance between the two cameras by photo recapture time. This was used to determine the use of the corridor by elephant species as either a corridor link or an habitat extension. The result showed that the uses of the corridor varied spatially and temporally. Some parts of the corridor were used as an habitat extension and other parts were used primarily as passage (Green et al, 2018).

Metapopulation theory is a theoretical framework that can be used to assess the long-term effect of habitat loss and fragmentation on species due to insufficient data for evidence based study (Ovaskainen & Hanski, 2003). It enables the understanding of how species react to fragmentation and is part of spatial population viability analysis (Ovaskainen & Hanski, 2003). Most metapopulation models predict that a population will become extinct after a critical value is reached (Hanski, 1999).

### 2.3 Mathematical Models of Metapopulation and Connectivity

Metapopulation was first modeled by Levins in 1969. His model assumes that the patches are identical, equally connected and infinite. The equilibrium fraction of occupied patches \( p^* \) is related to the extinction rate \( e \) of occupied patches and colonization rate, \( c \), of unoccupied patches by

\[
p^* = 1 - \frac{e}{hc} = 1 - \frac{\delta}{h}.
\]  

(2.1)
where $\delta = \frac{e}{c}$ and $h$ is the amount of suitable habitat patch for occupancy. Reduction in patch area increases $e$ and decreases $c$. Therefore, if $p^* > 0$ (that is $h > \delta$), the species will continue to survive (Hanski, 1999, Hanski, & Ovaskianen, 2003).

$$h > \delta.$$  \hfill (2.2)

Equation 2.2 implies that a species will begin to go extinct once the landscapes attribute, $h$, is less than the species attribute, $\delta$.

The rate of change in the fraction of occupied patches is

$$\frac{dp}{dt} = cp(1 - p) - ep.$$ \hfill (2.3)

However, the Levins model above cannot be applied to real metapopulation because of the unrealistic assumption of infinite, identical and equally connected landscapes. Therefore, Ovaskainen and Hanski (2003) extended the model to a Spatially Realistic Levins Model (SRLM). It is applicable to heterogeneous patch networks and models the probability that each patch is occupied. It is defined as:

$$\frac{d p_i}{d t} = C_i(p)(1 - p_i) - E_i(p)p_i,$$ \hfill (2.4)

where $C_i(p)$ is the colonization rate of patch $i$ and $E_i(p)$ is the extinction rate of patch $i$, and they both depend on the network spatial configuration. It is assumed that $E_i(p) = \frac{e}{A_i}$, where $\zeta$ is the effect of patch area on extinction risk, $e$ is the species-specific extinction rate and $A_i$ is the area of patch $i$. Also, the colonization rate of patch $i$ is assumed to be $C_i(p) = \sum_{j \neq i} C_{ij}p_j$, where $C_{ij}$ is the contribution of occupied patch $j$ to colonization of empty patch $i$. It is represented as $C_{ij} = (A_i)^{im}(A_j)^{em}f(d_{ij})p_j$. In addition, the Minimum Viable Metapopulation can be modeled as $p\sqrt{h} \geq 3$ (Hanski, & Ovaskianen, 2000, Hanski, & Ovaskianen, 2003, Hanski, 1999).
The metapopulation is represented by a matrix, $M$, such that $m_{ii} = 0$ and

$$m_{ij} = (A_i)^{im+ex}(A_j)^{em}f(d_{ij}).$$

The function, $f(d_{ij})$, is the kernel and often represented as $e^{-\alpha d_{ij}}$ where $\frac{1}{\alpha}$ is the average migration distance and $d_{ij}$ is the distance between patch $i$ and $j$. It was shown that if the leading eigenvalue of $M$, $\lambda_M$, is greater than $\delta$, then $p_i^* > 0$ for all $i$. Therefore, a species will persist in a given landscape if and only if

$$\lambda_M > \delta.$$  

(2.5)

The *Metapopulation capacity* of fragmented landscape is represented by $\lambda_M$. It represents the habitat amount in the non-spatial Levin’s model. Therefore, it includes the effect of spatial configuration (with habitat amount) on metapopulation persistence. Weighted average of $p_i^*$ was approximated to $p_i^* = 1 - \frac{\delta}{\lambda_M}$, which is similar to Levins’ model in equation (2.1). The SRLM was tested and verified on the Glanville fritillary butterfly (Hanski, & Ovaskianen, 2000, Hanski, & Ovaskianen, 2003).

Hanski and Ovaskianen (2003) also acknowledge that SPOM (Stochastic Patch Ocupancy Model) is a good method that assumes heterogeneous space. However, it has the disadvantage that it only models focal species presence or absence in a patch. It uses a Markov process with $2^n$ states for $n$ patches. A deterministic approximation of SPOM will result in the Levin model above (Hanski, & Ovaskianen, 2003).

Although the Metapopulation capacity describes the viability of the whole metapopulation, the contribution of an individual patch to the metapopulation dynamics is also necessary. This is helpful to identify the patches that must be conserved in order to enhance conservation when resources are scarce. Hanski and Ovaskainen (2003) developed a mathematical model for patch value with four definitions. First, patch value is the contribution of a patch to metapopulation capacity. It is the reduction in $\lambda_M$ when
the patch is removed. Secondly, it is the contribution of a patch to the metapopulation size. It is the decline in fraction of occupied patches after the removal of a patch. Thirdly, it is the reduction in metapopulation extinction mean time after patch removal (contribution to metapopulation persistence). Lastly, it was defined as a patch’s contribution to colonization events in the network. They are the left eigenvectors of $\lambda = 1$ in an appropriate transition matrix (Ovaskainen & Hanski, 2003).

A matrix of patch contribution to network colonization is represented by $B$. Each entry of matrix $B$ is defined as

$$b_{ij} = \frac{p_j c_{ij}}{\sum_{k \neq i} p_k c_{ik}},$$

(2.6)

where $b_{ij}$ is the direct contribution of patch $j$ to the colonization of patch $i$. This is a Markov chain since each entry of the matrix is non-negative, less than or equal to 1 and each row sums to 1. Therefore, the maximum eigenvalue is 1 and its left eigenvector, $W$, is the contribution of patch $j$ to the colonization event in the whole network. It enables a concise description of heterogenous metapopulation size as one number $p_\lambda$, which is the weighted fraction of occupied patch ($p_\lambda = \sum W_i p_i$) making the development of metapopulation theory easier (Ovaskainen & Hanski, 2003).

Alternative equilibria may also exist as a result of rescue effects or allee effects. Rescue effect decreases a population’s extinction rate by immigration. It changes the extinction rate to $\frac{e}{1+\sqrt{cp}}$. Allee effect reduces the colonization rate from $cp$ to $cp^2$ when the state of occupancy of other metapopulation patches is low (Ovaskainen & Hanski, 2003).

Andersson & Bodin (2008) complemented the metapopulation dynamics by focusing on how spatial arrangement of habitat patches affects the value of an habitat,
which will be helpful in land planning of heavily fragmented habitat. Therefore, they
used a \textit{network-centric model} to assess landscape connectivity threshold for target
species, even when limited data is available. The landscape is represented as a network
with a vertex representing each habitat patch. An edge connects a pair of vertices (patch)
if exchange of species can take place between the two patches. This is used to simulate
the suitability of a landscape to different species. The graph theoretic approach was tested
using empirical data from Stockholm, Sweden urban landscape. The authors concluded
that a collection of well-connected small habitat patches can be sufficient for species
survival. In addition, the authors found that the matrix nature plays a significant role on
connectivity and species survival (Andersson & Bodin, 2008).

Telemetry dataset from GPS radio collar on species has been a very common
method used to gather data on species movement. Two commonly used movement model
are the continuous time Markov chain (CTMC) and the step selection functions model
(SSFs). The continuous time Markov chain Model was first developed by Hanks et al.
(2015). The authors transformed GPS dataset on species location to a compressed
continuous-time discrete space path on surfaces that has been gridded. The step selection
function model is a special case of resource selection model that was first used by Fortin
et al. (2005) to model elk movement. Characteristic of observed movement and random
possible movement of elk at 5 hours interval was compared and used to model elk
movement pattern based on environmental attributes (Fortin et al., 2005). Although,
Brennan et al. (2018) showed that connectivity model using CTMC gave a better forecast
of elk movement than models using SSF, this may vary across different species.

Clark et al (2015) used step selection function to measure movement of Louisiana
black bears between four habitat patches. The authors used GPS collar dataset of 8 female and 23 male bears and also determined the characteristics of the travel route. Step function model was developed by comparing chosen bear step to possible random steps. The model was used to create 4000 hypothetical correlated random walk for each patch and the proportion that intersected another patch was estimated. This proportion was used to quantify the effectiveness of the connectivity or corridor (Clark et al., 2015). Recall that this is similar to the definition of transition probability by Tischendorf and Wissel (1997).

The use of circuit theory to model connectivity in a landscape was introduced by McRae et al. (2008). It has been used extensively to determine and model corridors. For instance, Gantchoff and Belant (2017) use it to determine areas that serves as corridors for connectivity of American Black bears. In addition, Brennan et al., (2018) concluded that the circuit theory gave a better estimate than Shortest Random Path (SRP).

### 2.4 Problem Definition

The application goal of this dissertation is to create a mathematical structure that can estimate and determine a quantity (*connectivity*) that will define the effectiveness of corridors in connecting fragmented habitat patches. Given a set of habitat patches, how can they be efficiently connected to enhance the survival of species? What qualities should the model quantity have to be able to achieve the required corridor connectivity and supplementation? Although Clark, et al (2015) attempted to answer some of these questions by a step function simulation model of black bears, we will be using the eigenvalues and other introduced measures to quantify connectivity in this dissertation. The following steps will be required.
Step 1: Create a digraph model of the landscape (with corridor) and its adjacency matrix.

Step 2: Transform the adjacency matrix to a probability matrix by dividing each row by its row sum. Steps 1 and 2 could also be combined by forming a probability matrix directly by applying techniques used by Tischendorf and Wissel (1997) and Clark, et al (2015). This will be discussed in Chapter 5.

Step 3: Evaluate eigenvalues and left leading eigenvectors.

Step 4: Evaluate other measures of the system’s dynamic. These measures will be introduced and defined in Chapter 3 of this dissertation.

Step 5: Determine the effectiveness of corridor or landscape connectivity.
Chapter 3
CHARACTERIZATION OF MARKOV CHAIN SPECTRUM

This Chapter introduced and defined new, original and significant concepts and theorems that could have applications across different disciplines, beyond ecology. Some of the theorems were completely proved mathematically and some are left as conjectures, because a complete mathematical proof is yet to be provided. Some of the new and original concepts developed in this chapter are homoprobability, coperiodic weight, coperiodic cospectral and the dominant set of spectral loop/cycles. First passage time, which is a well-known concept, was discussed in Section 3.5 and used to develop and discuss relationship with some of the newly introduced concepts.

3.1 Properties of Markov Chain Eigenvalues

As stated in Chapter 1, a Markov chain can be represented by its transition (or probability matrix). The transition matrix of any n states chain has n eigenvalues which are all on or inside the unit circle in the complex plane. Hence, the maximum eigenvalue is 1 and it is always an eigenvalue. If 1 is the only eigenvalue of modulus 1, then a unique limit distribution exists. If a unique limit distribution has nonzero components, then every sample path will describe the same transition matrix (see Chapter 4.4 and Merrill, 2010). The rate and mode of sample path approach to the limit distribution can also be describe by the eigenvalues other than 1 (Merrill, 2010).
3.2 The Eigenvalues of 2-state Markov Chain

As stated earlier, 1 is always an eigenvalue of a Markov chain. From Theorem 1.2.1(i), the trace of any matrix is the sum of the eigenvalues. Therefore, the eigenvalues of any two state Markov chain $M$ with transition matrix, $P$, are:

$$\lambda = 1 \text{ and } \lambda = (\text{trace}(P) - 1).$$

Observe that the $\text{trace}(P)$ is the sum of the self-loops of $M$. Since the determinant of a matrix is the product of the elements of the spectrum (eigenvalues), then $\text{det}(P)$ is also $\text{trace}(P) - 1$. Thus, if the sum of self-loop is less than 1, $\text{det}(P)$ and second eigenvalue are both negative. Determinant of $P$ and second eigenvalues are zero, when the self-loops sum to one. And they are positive when self-loop sum is greater than 1.

3.3 Characterization of Eigenvalues of 3-state Markov Chain

A 3 state Markov chain may be represented by the stochastic matrix, $M$, below such that $a + d + i = 1$, $e + b + g = 1$ and $h + f + c = 1$. Note that, $M$ may also be used to represent stochastic matrix from this point in the dissertation.

$$M = \begin{bmatrix} a & d & i \\ e & b & g \\ h & f & c \end{bmatrix}$$

Figure 3a: 3-state Markov chain
Since the $\sum \lambda = \text{trace}(M)$. The trace of the probability matrix has an effect (necessary, but not sufficient) on the nature of the eigenvalues. We will consider the eigenvalues of general 3-state Markov chains first, and then look at the eigenvalues of some specific 3-state chains.

### 3.3.1 General case for 3-state Markov chain

**Theorem 3.1** Let $D$ be the $\det(M)$ and $T = \frac{\text{Trace}(M) - 1}{2}$ of a 3-state stochastic digraph, $M$. Then the eigenvalues of $M$ are:

$$1, T \pm \sqrt{T^2 - D},$$

Thus, the eigenvalues other than 1 are equidistance from $T$.

**Proof**

Let $M = \begin{bmatrix} a & d & i \\ e & b & g \\ h & f & c \end{bmatrix}$ be the stochastic matrix of a 3 state Markov chain.

Also, let $t$ be $\text{trace}(M)$, and $D$ be $\det(M)$.

Then $t = a + b + c$, $D = abc - (afg + bhi + cde) + dgh + ife$ and $T = \frac{t - 1}{2}$.

The characteristic equation is

$$(a - \lambda)(b - \lambda)(c - \lambda) - (a - \lambda)fg - (b - \lambda)hi - (c - \lambda)de + dgh + ife = 0 \quad (3.1)$$

The LHS of (3.1) can be written as

$$-\lambda^3 + (a + b + c)\lambda^2 - (ab + bc + ac)\lambda + (hi + fg + de)\lambda + abc - afg - bhi - cde + dgh + ife$$

$$= -\lambda^3 + (t)\lambda^2 - (ab + bc + ac)\lambda + (hi + fg + de)\lambda + D$$

$$= -\lambda^3 + (t)\lambda^2 + (hi + fg + de - ab - bc - ac)\lambda + D.$$ 

Hence, the characteristic equation can be written as
\[-\lambda^3 + (t)\lambda^2 + (hi + fg + de - ab - bc - ac)\lambda + D = 0.\]

Since \(\lambda = 1\) is a solution, substituting \(\lambda = 1\) in the characteristic equation gives,

\[-1 + t + hi + fg + de - ab - bc - ac + D = 0\]

Hence, \(1 - t - D = hi + fg + de - ab - bc - ac\)

Hence, the characteristic polynomial becomes

\[-\lambda^3 + (t)\lambda^2 + (1 - t - D)\lambda + D = 0\]

Thus, the eigenvalues are, \(\lambda = 1\) and the solution of \((\lambda^2 + (1 - t)\lambda + D)\) a quadratic equation. Using the quadratic formula to solve for \((\lambda^2 + (1 - t)\lambda + D) = 0\), we have

\[\lambda = \frac{-(1-t) \pm \sqrt{(t-1)^2 - 4D}}{2}\]

\[= \frac{- (1-t)}{2} \pm \sqrt{\frac{(t-1)^2 - 4D}{4}}\]

\[= \frac{t-1}{2} \pm \sqrt{\left(\frac{(t-1)^2}{2^2}\right) - D}\]

\[= T \pm \sqrt{T^2 - D}\]

Thus, the eigenvalues are

\[\lambda = 1, T \pm \sqrt{T^2 - D}\]

Alternatively, we can prove Theorem 3.1 by showing that \(1 + \lambda_2 + \lambda_3 = \text{trace}(M)\) and \(1 \ast \lambda_2 \ast \lambda_3 = D\) as shown below.

\[\text{Alternative proof:} \quad 1 + \lambda_2 + \lambda_3 = 1 + T + \sqrt{T^2 - D} + T - \sqrt{T^2 - D}\]

\[= 1 + 2T\]

\[= 1 + 2\left(\frac{\text{Trace}(P) - 1}{2}\right)\]

\[= \text{Trace}(M).\]
Also,

\[ 1 \ast \lambda_2 \ast \lambda_3 = 1 \ast (T + \sqrt{T^2 - D}) \ast (T - \sqrt{T^2 - D}) = T^2 - (T^2 - D) = D. \]

**Theorem 3.2** In a 3-state irreducible stochastic digraph \( M \), the eigenvalues, other than \( \lambda = 1 \), have the following 3 possible properties:

(i) The eigenvalues, \( \lambda_2 \) and \( \lambda_3 \), are real and have opposite signs, if and only if \( D < 0 \).

(ii) The eigenvalues, \( \lambda_2 \) and \( \lambda_3 \), are real and have the same sign if and only if \( 0 < D < T^2 \) (depending on \( T \) or \( \text{trace}(M) \)).

- If \( \text{trace}(M) < 1 \), then \( \lambda_1 = 1, \lambda_2 < 0 \) and \( \lambda_3 < 0 \)
- If \( \text{trace}(M) > 1 \), then \( \lambda_1 = 1, \lambda_2 > 0 \) and \( \lambda_3 > 0 \).

(iii) The eigenvalues, \( \lambda_2 \) and \( \lambda_3 \), are complex conjugates if and only if \( D > T^2 \).

This implies that the value of \( D \) in relation to \( T^2 \) can give information about the nature of the eigenvalues of a 3-state Markov chain.

**Proof**

(i) Note that determinant of \( M \), denoted as \( D \), is always a real number for any 3-state Markov chain.

We have 2 parts to this proof.

**Part 1:** Suppose the eigenvalues have real values with nature \( \lambda_1 = 1, \lambda_2 < 0 \) and \( \lambda_3 > 0 \). We want to show that \( D < 0 \).

By Theorem 1.2.1(ii) the product of all the eigenvalues is the \( \text{det} (M) \). Hence,

\[ D = \lambda_1 \ast \lambda_2 \ast \lambda_3 \]
\[= 1 \times \lambda_2 \times \lambda_3\]

\[< 0 \text{ (since, } \lambda_2 < 0, 1 > 0 \text{ and } \lambda_3 > 0)\]

Thus, \(D < 0\).

**Part 2:** Suppose \(D < 0\). We want to prove that the eigenvalues are real and have the following natures: \(\lambda_1 = 1, \lambda_2 < 0\) and \(\lambda_3 > 0\) (or \(\lambda_2 > 0\) and \(\lambda_3 < 0\)).

We have 3 cases depending on the value of \(T\) or \(\text{trace}(M)\).

**Case 1:** \(T = 0\)

By Theorem 3.1 the eigenvalues are,

\[\lambda = 1, T \pm \sqrt{T^2 - D}\]

\[= 1, \pm \sqrt{-D} \text{ (because } T = 0)\]

\[= 1, \pm \sqrt{|D|} \text{ since } D < 0\]

\[= 1, x, -x \text{ for real number } x = \sqrt{|D|} \text{ and } x > 0.\]

Therefore, \(\lambda_1 = 1, \lambda_2 < 0\) and \(\lambda_3 > 0\).

**Case 2:** \(T < 0\)

By Theorem 3.1, the eigenvalues are,

\[\lambda = 1, T \pm \sqrt{T^2 - D}\]

\[= 1, T \pm \sqrt{T^2 + |D|} \text{ since } D < 0.\]

For \(x, y > 0\), let \(T = -x\) and \(\sqrt{T^2 + |D|} = \pm y\).

Thus, \(\lambda = 1, -x \pm y\) by Theorem 3.1.

Hence, there exists \(\lambda_2 = -x - y < 0\) because \(x\) and \(y > 0\).

To show that \(-x + y > 0\), observe that \(T^2 + |D| > T^2\). Hence, \(y^2 > x^2\). This implies that \(y > x\) since \(x, y > 0\).
Hence, $\lambda_3 = -x + y > 0$.

Thus, $\lambda_1 = 1, \lambda_2 < 0$ and $\lambda_3 > 0$.

**Case 3: $T > 0$**

By Theorem 3.1, the eigenvalues are,

$$
\lambda = 1, T \pm \sqrt{T^2 - D} \\
= 1, T \pm \sqrt{T^2 + |D|} \text{ since } D < 0.
$$

For $x, y > 0$, let $T = x$ and $\sqrt{T^2 + |D|} = \pm y$.

Thus, $\lambda = 1, x \pm y$.

Observe that $\lambda_3 = x + y > 0$ since $x > 0$ and $y > 0$.

To show that $\lambda_2 = x - y < 0$, observe that $T^2 + |D| > T^2$.

Hence,

$$
y^2 > x^2 \\
y > x.
$$

Thus, $x - y < 0$.

Therefore, $\lambda_1 = 1, \lambda_2 < 0$ and $\lambda_3 > 0$.

Alternatively, we may prove this theorem in a shorter form as follow.

**Alternative Proof**

By Theorem 3.1, $\lambda = 1, T \pm \sqrt{T^2 - D}$.

$$
\lambda = 1, T \pm \sqrt{T^2 + |D|} \text{ since } D < 0.
$$

Since $T$ and $\sqrt{T^2 - D}$ are real, $\lambda = 1, T \pm \sqrt{T^2 - D}$ are also real.

Next, we show that $\lambda$s have natures, $\lambda_1 = 1, \lambda_2 < 0$ and $\lambda_3 > 0$.

Let $\sqrt{T^2 + |D|} = \pm y$. Then $y > |T|$ since $T^2 + |D| > T^2$. 
Now, we have 3 cases for $\lambda_3 = T + y$ and $\lambda_2 = T - y$.

Case 1: if $T > 0$, then $\lambda_3 = T + y > 0$ and $\lambda_2 = T - y < 0$ (because $y > T$).

Case 2: if $T = 0$, then $\lambda_3 = T + y > 0$ and $\lambda_2 = T - y < 0$.

Case 3: if $T < 0$, then $\lambda_3 = T + y > 0$ since $y > T$ and $\lambda_2 = T - y < 0$.

Thus, $\lambda_1 = 1, \lambda_2 < 0$ and $\lambda_3 > 0$.

(ii)

Part 1: Suppose $0 < D < T^2$ for a 3-state Markov chain. We want to prove that $\lambda_2$ and $\lambda_3$ are either both positive or both negative (same side of 0 on the real number line).

By Theorem 3.1, $\lambda_2$ and $\lambda_3$ are $T \pm \sqrt{T^2 - D}$.

Let $y = \sqrt{T^2 - D}$ and $y > 0$. Observe that $T^2 - D < T^2$ since $0 < D < T^2$. Thus,

$$y < |T|.$$  \hspace{1cm} (3-2)

Now, $\lambda_2 = T + y$ and $\lambda_3 = T - y$. And we have two cases based on the value of $T$.

**Case 1:** Suppose $T < 0$

Then, $\lambda_2 = T + y < 0$ since $y < |T|$. Also, $\lambda_3 = T - y < 0$.

Thus, $\lambda_2$ and $\lambda_3$ are both negative.

**Case 2:** $T > 0$

Then, $\lambda_2 = T + y > 0$. And $\lambda_3 = T - y > 0$, since $y < T$.

Thus, $\lambda_2$ and $\lambda_3$ are both positive.

Therefore, $\lambda_2$ and $\lambda_3$ are on the same sides of 0 on the real number line.

Part 2:

Suppose $\lambda_2$ and $\lambda_3$ are real and have the same sign (on the same side of 0 on the real number line), then $0 < D < T^2$. By Theorem 1.2.1(ii), $D = 1 \ast \lambda_2 \ast \lambda_3$. Hence, $D > 0$ since $\lambda_2$ and $\lambda_3$ have same sign.
Next, we show that $D < T^2$. By way of contradiction, suppose $D > T^2$. Then

$T^2 - D < 0$ and $\lambda_2$ and $\lambda_3$ will be complex. This is a contradiction to the statement that $\lambda_2$ and $\lambda_3$ are both real. Thus, $D < T^2$.

Therefore, $0 < D < T^2$.

(iii)

First, we show that if $D > T^2$ then $\lambda = T \pm (y)i$ (complex conjugate).

Suppose $D > T^2$. We already showed that $\lambda_2$ and $\lambda_3$ are $T \pm \sqrt{T^2 - D}$.

Observe that $T^2 - D < 0$ since $D > T^2$. Thus, $\sqrt{T^2 - D} = (y)i$.

And $\lambda = T \pm (y)i$ are complex conjugates.

Next, we show that if $\lambda = T \pm (y)i$ (complex conjugate), then $D > T^2$.

Observe that $D = \lambda_1 * \lambda_2 * \lambda_3$. Hence,

$$D = 1. (T + yi). (T - yi)$$

$$= T^2 - y^2$$

$$= T^2 + y^2$$

$$> T^2.$$ 

Therefore, $D > T^2$. $\blacksquare$

3.3.2 Zero Trace

This is when $\text{trace}(M) = 0$. That is, $a = b = c = 0$ in the stochastic matrix $M$.

This has application in metapopulation ecology since the diagonal entries are zeros. The landscape matrices model used for computing the metapopulation capacity and patch value by contribution to colonization has no self-loops. This is because of the assumption that an empty patch cannot colonize itself.
\[
M = \begin{bmatrix}
0 & d & i \\
e & 0 & g \\
h & f & 0
\end{bmatrix}
\]

Observe that the determinant of the matrix is the sum the two period 3 loops

\( z = dgh + ife \) and \( T^2 = 0.25 \). Thus,

- If \( z = 0.25 \), the eigenvalues are: \( \lambda = 1, -0.5, -0.5 \)
- If \( z > 0.25 \), the eigenvalues are \( \lambda = 1, -0.5 \pm yi \) with \( y = \sqrt{0.25 - z} \)
- If \( z < 0.25 \), the eigenvalues are real and \( \lambda = -0.5 \pm y \) with \( y = \sqrt{0.25 - z} \)

Thus, \( \lambda = 1, -0.5 \pm \sqrt{0.25 - z} \) where \( z \) is the sum of the two possible period 3 loops, \( z = dgh + ife \). Observe that the only possible eigenvalue nature aside from \( \lambda = 1 \) are complex and real negatives eigenvalues. A zero eigenvalue is only possible when \( z = 0 \).

### 3.3.3 Homoprobability Markov Chain

A Homoprobability Markov chain is one that has the same probability for each state. That is, probability of remaining in a state, moving left, or right is the same for each states in the Markov chain. This can be specifically called loop homoprobability. This will be described below with the other one, which is the line homoprobability.

#### Loop homoprobability

For a 3-state Markov chain, the corresponding digraph has a pair of simple cycles on the three vertices. The digraph and transition matrix are below.

\[
\begin{bmatrix}
a & 1 - a - b & b \\
b & 1 - a - b & a \\
1 - a - b & b & a
\end{bmatrix}
\]
Figure 3b: Loop homoprobability Markov chain

The eigenvalues of this stochastic digraph are always on or inside the equilateral triangle inscribed in the unit circle whose vertices are on the cube roots of unity. The eigenvalues are complex, except when \( a + 2b = 1 \).

Figure 3c: Equilateral triangle inscribed in the unit circle, whose vertices are on the cube root of unity

**Theorem 3.3:** The eigenvalues of a loop homoprobability Markov chain are:

\[
1, \frac{3a-1}{2} \pm \sqrt{0.75} (1 - a - 2b)i.
\]

**Proof**

Observe that trace(M), \( t = 3a \), \( T = \frac{3a-1}{2} \) and

\[
D = a^3 - 3ab(1-a-b) + (1-a-b)^3 + b^3.
\]

By Theorem 3.1, the eigenvalues are \( 1, T \pm \sqrt{T^2 - D} \).

\[
\sqrt{T^2 - D} = \sqrt{\frac{(3a-1)^2}{4} - a^3 + 3ab(1-a-b) - (1-a-b)^3 - b^3}
\]
\[
\sqrt{T^2 - D} = \sqrt{\frac{9a^2}{4} - \frac{6a}{4} + \frac{1}{4} - a^3 + 3ab(1 - a - b) - (1 - a - b)^3 - b^3}
\]

Observe that \((1 - a - b)^3 = 1 - 3a - 3b + 6ab + 3a^2 + 3b^2 - a^3 - 3a^2b - 3ab^2 - b^3\).

Thus,
\[
\sqrt{T^2 - D} = \sqrt{\frac{9a^2}{4} - \frac{6a}{4} + \frac{1}{4} - 1 - 3ab + 3a + 3b - 3a^2 - 3b^2}
\]
\[
= \sqrt{-0.75 + 6a - 3ab + 3b - \frac{3}{4}a^2 - 3b^2}
\]
\[
= \sqrt{-0.75 (1 + 4ab - 2a - 4b + a^2 + 4b^2)}
\]
\[
= \sqrt{-0.75(1 - a - 2b)^2}
\]
since \((1 - a - 2b)^2 = 1 - 2a - 4b + a^2 + 4ab + 4b^2\).

Since \(T = \frac{3a - 1}{2}\) and \(\sqrt{T^2 - D} = \sqrt{-0.75(1 - a - 2b)^2}\)
\[
= \sqrt{0.75(1 - a - 2b)i}.
\]

The eigenvalues are,
\[
T \pm \sqrt{T^2 - D} = \frac{3a - 1}{2} \pm \sqrt{0.75(1 - a - 2b)i}.
\]

Thus, the eigenvalues of the loop homoprobability Markov chain are:
\[
1, \frac{3a - 1}{2} \pm \sqrt{0.75(1 - a - 2b)i}.
\]

**Theorem 3.4:** The eigenvalues of any 3-state, loop homoprobability Markov chain are on or inside the equilateral triangle inscribed in the unit circle with triangle vertices on the 3 cube roots of unity.

**Proof**

Observe that the equilateral triangle whose vertices are on the cube roots of unity is bounded by the following inequalities in the complex plane.
\[ x \geq -0.5 \]
\[ y \geq \frac{2(\sqrt{0.75})}{3} (x - 1) \text{ and} \]
\[ y \leq -\frac{2(\sqrt{0.75})}{3} (x - 1) \]

Note that \( \frac{2(\sqrt{0.75})}{3} \approx \pm 0.5773 \).

Here \( x \) is the real axis and \( y \) is the imaginary axis of the complex plane.

It remains to show that the eigenvalues of a 3-state homoprobability Markov chain are bounded by these 3 inequalities. Let the real part of the eigenvalues be \( x = \frac{3a-1}{2} \).

Observe that \( a \) is bounded by 0 and 1. That is, \( 0 \leq a \leq 1 \). Substituting this range of \( a \) in \( x \), we have, \( -0.5 \leq x \leq 1 \). Hence, -0.5 is the least lower bound for \( x \). Thus,

\[ x \geq -0.5. \]

Since \( x = \frac{3a-1}{2} \) then, \( a = \frac{2x+1}{3} \).

To prove the remaining two inequalities, we have 3 cases based on the values of \( b \)

**Case 1**: Let \( b = 0 \)

Then,

\[ y = \sqrt{0.75}(1 - a - 2b) \]
\[ = \sqrt{0.75}(1 - a) \]
\[ = \sqrt{0.75}(1 - \left(\frac{2x+1}{3}\right)) \]
\[ = \sqrt{0.75}\left(\frac{3-2x-1}{3}\right) \]
\[ = \sqrt{0.75}\left(\frac{2-2x}{3}\right) \]
\[ = -\frac{2(\sqrt{0.75})}{3} (x - 1). \]

Thus,
\[ y = \pm \frac{2(\sqrt{0.75})}{3} (x - 1). \]

**Case 2:** Let \( b = 1 - a \)

Then,

\[ y = \sqrt{0.75}(1 - a - 2b) \]
\[ = \sqrt{0.75}(b - 2b) \]
\[ = \sqrt{0.75}(-b) \]
\[ = \sqrt{0.75}(a - 1) \]
\[ = \sqrt{0.75}\left(\frac{2x+1}{3} - 1\right) \]
\[ = \sqrt{0.75}\left(\frac{2x+1-3}{3}\right) \]
\[ = \sqrt{0.75}\left(\frac{2x-2}{3}\right) \]
\[ = \frac{2(\sqrt{0.75})}{3} (x - 1). \]

Thus,

\[ y = \pm \frac{2(\sqrt{0.75})}{3} (x - 1). \]

**Case 3:** Let \( 0 < b < 1 - a \)

Then,

\[ y = \sqrt{0.75}(1 - a - 2b) \]
\[ = \sqrt{0.75}\left(\frac{2-2x}{3} - 2b\right) \]
\[ = \sqrt{0.75}\left(\frac{2-2x-6b}{3}\right) \]
\[ = \frac{2(\sqrt{0.75})}{3} (1 - x - 3b). \]

Now,
\[ b < 1 - a \]
\[ = 1 - \frac{2x+1}{3} \]
\[ = \frac{3-2x-1}{3} \]
\[ = \frac{2-2x}{3}. \]

Hence, \( 3b < 2 - 2x \). This implies that

\[ -3b > 2x - 2. \]

Adding \( 1 - x \) to both sides gives \( 1 - x - 3b > x - 1 \).

Substituting in equation 3.1 above gives

\[ y = \frac{2(\sqrt{0.75})}{3}(1 - x - 3b) > \frac{2(\sqrt{0.75})}{3}(x - 1). \]

Thus, \( y > \frac{2(\sqrt{0.75})}{3}(x - 1). \)

**Line homoprobability**

Line Homoprobability is when the 3-state homoprobability chain has no cycle on 3 nodes. The matrices and weighted digraphs are always in the form below.

\[
\begin{bmatrix}
  a + b & 1 - a - b & 0 \\
  b & a & 1 - a - b \\
  0 & b & 1 - b \\
\end{bmatrix}
\]

**Figure 3d:** Line Homoprobabilistic Markov chain
**Theorem 3.5:** The eigenvalues of a line homoprobability chain are:

\[ \lambda = 1, a \pm \sqrt{b(1-a-b)}. \]

**Proof**

Observe that \( T = a \), because, \( \frac{(a+b+a+1-b-1)}{2} = \frac{2a}{2} = a \).

Also,

\[
D = (a + b)(a)(1 - b) - [(a + b)(b)(1 - a - b) + (1 - b)(b)(1 - a - b)] \\
= (a + b)(a)(1 - b) - [(b)(1 - a - b)(a + b + 1 - b)] \\
= (a + b)(a)(1 - b) - [(b)(1 - a - b)(1 + a)] \\
= a^2 + ab - a^2 b - ab^2 - [b - ab - b^2 + ab - a^2 b - ab^2] \\
= a^2 - b + ab + b^2 \\
= a^2 - b(1 - a - b).
\]

Thus,

\[
T^2 - D = a^2 - (a^2 - b(1 - a - b)) \\
= b(1 - a - b).
\]

By Theorem 3.1, the eigenvalues \( \lambda = 1, T \pm \sqrt{T^2 - D} \), are

\[ \lambda = 1, a \pm \sqrt{b(1-a-b)}. \]

**3.4 Other Characterizations of Markov Chain Spectrum**

Other observations about the eigenvalues of reducible Markov chains are as follows.

1. The eigenvalue of an isolated vertex is 1.

2. The multiplicity of \( \lambda = 1 \) is the number of recurrent strong components (communicating class) in a Markov chain.
3. The eigenvalues of a strong component on one vertex is the weight of its self-loop.

4. If there exists a transient class with only one vertex, then the eigenvalue for the class will be the weight of the self-loop. Therefore, \( \lambda = 0 \) if it has no self-loop.

### 3.5 First Passage Time and Measure

The first passage time from vertex \( i \) to \( j \) is the minimum time required to move from state \( i \) to state \( j \) in any time step \( n \). It is denoted below (Heyman and Sobel, 2004).

\[
T_{ij} = \inf \{ t : X(t) = j | X(0) = i \} \quad i \neq j
\]

\( T_{ii} \) is the first return time and \( \inf \) is infimum.

Let \( f^{(n)}_{ij} \) be the first passage time probability. If the Markov chain is recurrent, the set \( \{f^{(n)}_{ij}\}_{n=0}^{\infty} \) will define the probability distribution for random variable, \( T_{ij} \). Hence, the mean first passage time, \( m_{ij} = E(T_{ij}) \), will be as shown below (Allen, 2003).

\[
m_{ij} = \sum_{n=1}^{\infty} n f^{(n)}_{ij}
\]

**Definition 3.5.16** (Heyman and Sobel, 2004). The mean first passage time is

\[
m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}.
\]

Then, \( m_{ij} = 1 \) if transition from state \( i \) to \( j \) is in one step (that is \( k = j \)).

Otherwise, \( m_{ij} = n \) (for integer, \( n > 1 \)) if the first transition is to a state \( k \neq j \) and the first passage time from \( k \) to \( j \) is in \( n - 1 \) steps (Heyman and Sobel, 2004).
The n-step transition probabilities and the first return probabilities are not the same except for \( n = 1 \). That is, \( p_{11} = f_{11} \). The relationship between them is defined (in Allen, 2003) as

\[
p^{(n)}_{ii} = \sum_{k=1}^{n} f^{(k)}_{ii} p^{n-k}_{ii}
\]  

(3.2)

where \( p^{(0)}_{ii} = 1 \). Similarly,

\[
p^{(n)}_{ij} = \sum_{k=1}^{n} f^{(k)}_{ij} p^{n-k}_{ij}.
\]

Hence, we can say that, for an n-state chain, the total probability that first return time is 1 is \( \sum_{i=1}^{n} f^{(1)}_{ii} \). We will call this loop 1 total weight, \( L_1 \). Similarly, the loop 2 total weight, \( L_2 \), will be \( \sum_{i=1}^{n} f^{(2)}_{ii} \) and loop 3 total weight, \( L_3 \), is \( \sum_{i=1}^{n} f^{(3)}_{ii} \). For a 3-state Markov chain,

\[
L_1 = f^{(1)}_{11} + f^{(1)}_{22} + f^{(1)}_{33}.
\]

\[
L_2 = f^{(2)}_{11} + f^{(2)}_{22} + f^{(2)}_{33}.
\]

\[
L_3 = f^{(3)}_{11} + f^{(3)}_{22} + f^{(3)}_{33}.
\]

Using the relationship in equation 3.2 above,

\[
p^{(1)}_{ii} = f^{(1)}_{ii}.
\]

\[
p^{(2)}_{ii} = (p_{ii})^2 + f^{(2)}_{ii}.
\]

Thus,

\[
f^{(2)}_{ii} = p^{(2)}_{ii} - (p_{ii})^2.
\]

Similarly,

\[
p^{(3)}_{ii} = p_{ii}(p_{ii})^2 + f^{(2)}_{ii} p_{ii} + f^{(3)}_{ii}.
\]

Thus,
\[ f^{(3)}_{ii} = p^{(3)}_{ii} - p_{ii}(p_{ii})^2 - f^{(2)}_{ii}. \]

\[ = p^{(3)}_{ii} - (p_{ii})^3 - [ p^{(2)}_{ii} - (p_{ii})^2]p_{ii}. \]

\[ = p^{(3)}_{ii} - p_{ii}(p^{(2)}_{ii}). \]

Therefore,

\[ L_1 = p_{11} + p_{22} + p_{33} = \text{trace}(P) \]

\[ L_2 = [ p^{(2)}_{11} - (p_{11})^2 ] + [ p^{(2)}_{22} - (p_{22})^2 ] + [ p^{(2)}_{33} - (p_{33})^2 ] \]

\[ = [ p^{(2)}_{11} + p^{(2)}_{22} + p^{(2)}_{33} ] - [ (p_{11})^2 + (p_{22})^2 + (p_{33})^2 ] \]

\[ = [\text{trace}(P^2)] - [ (p_{11})^2 + (p_{22})^2 + (p_{33})^2 ] \]

\[ = [\text{trace}(P^2)] - \sum_{i=1}^{3} (p_{ii})^2. \]

Similarly,

\[ L_3 = [\text{trace}(P^3)] - \sum_{i=1}^{3} p_{ii}(p^{(2)}_{ii}) - \sum_{i=1}^{3} (p_{ii})^3. \]

**Figure 3e:** Line Markov chain

For any 3-states line Markov chain like Figure 3e and matrix above, where the end patches cannot be reached without going through the intermediate state, we show that the first passage time will be \( m_{13} = \frac{1 + p_{12} - p_{22}}{p_{12} * p_{23}} \) and \( m_{31} = \frac{1 + p_{32} - p_{22}}{p_{32} * p_{21}} \) respectively.

We will be using this repeatedly in our application in the Chapter 5.
Theorem 3.5.17 For the 3-state line Markov chain of the nature of Figure 3e above, the mean first passage times are:

\[ m_{13} = \frac{1 + p_{12} - p_{22}}{p_{12} + p_{23}} \quad \text{and} \]

\[ m_{31} = \frac{1 + p_{32} - p_{22}}{p_{32} + p_{21}} \]

**Proof**

Observe that \( m_{13} = 1 + \sum_{k \neq 3} p_{1k} m_{k3} \) by Definition 3.5.16.

Then,

\[ m_{13} = 1 + p_{11} m_{13} + p_{12} m_{23} \]

\[ (1 - p_{11}) m_{13} = 1 + p_{12} m_{23} \]

Note that,

\[ m_{23} = 1 + \sum_{k \neq 3} p_{2k} m_{k3} \]

\[ = 1 + p_{21} m_{13} + p_{22} m_{23} \]

\[ (1 - p_{22}) m_{23} = 1 + p_{21} m_{13} \]

\[ m_{23} = \frac{1 + p_{21} m_{13}}{(1 - p_{22})} \]

By substituting \( m_{23} = \frac{1 + p_{21} m_{13}}{(1 - p_{22})} \) in \( (1 - p_{11}) m_{13} = 1 + p_{12} m_{23} \) from above,

\[ (1 - p_{11}) m_{13} = 1 + p_{12} \left( \frac{1 + p_{21} m_{13}}{(1 - p_{22})} \right) \]

\[ (1 - p_{11}) m_{13} - \frac{p_{12} p_{21} m_{13}}{(1 - p_{22})} = 1 + \frac{p_{12}}{(1 - p_{22})} \]

\[ \frac{(1 - p_{22})(1 - p_{11}) m_{13} - p_{12} p_{21} m_{13}}{(1 - p_{22})} = 1 - p_{22} + \frac{p_{12}}{(1 - p_{22})} \]

\[ (1 - p_{22})(1 - p_{11}) m_{13} - p_{12} p_{21} m_{13} = 1 - p_{22} + p_{12} \]

\[ m_{13} = \frac{1 - p_{22} + p_{12}}{(1 - p_{22})(1 - p_{11}) - p_{12} p_{21}} \]
\[ m_{13} = \frac{1 p_{12} - p_{22}}{p_{12} \cdot p_{23}} \]

Therefore,

\[ m_{31} = \frac{1 p_{32} - p_{22}}{p_{32} \cdot p_{21}} \]

3.6 Eigenvalue Nature and Dominant Spectral Loop

Merrill (2010) stated that the nature of the eigenvalues gives information about the sample path approach to limit distribution. The sample path is discussed further in Chapter 4.4 (See Figure 4f). For instance, a negative eigenvalue can indicate a cycle of length 2 approach to limit distribution, a complex eigenvalue might indicate a cycle of length 3 or 4 approach to the limit distribution. The doubling periodicity of the discrete logistic equation was used to demonstrate this statement (Merrill, 2010).

3.6.1 Spectra Nature and Loop Structure

Consider the 3-state Markov chain in Figure 3a, and its stochastic matrix. This digraph has eight loops of which three of them are self-loops (cycles of length 1), three are cycles of length 2 and two are cycles of length 3. The loop weights of a weighted digraph are the product of the weight of each arc in a loop in the digraph. Thus, for a non-weighted digraph, the loop weights will be 1.
The elements of each cycle loop weight are:

Cycle 1 loop weight, $P_1$: \{a, b, c\}.

Cycle 2 loop weights, $P_2$: \{fg, hi, de\}.

Cycle 3 loop weights, $P_3$: \{ife, dgh\}.

Observe that these loop weights are related to loop total weights, $L$, introduced in Section 3.5 as follows:

$$L_1 = \sum P_1$$
$$L_2 = 2 \sum P_2$$
$$L_3 = 3 \sum P_3$$

where $\sum P_n$ is the sum of the elements in $P_n$ and $n$ is the length of the cycle.

Also, the determinant of a Markov chain, $D$, is a certain combination of the loop weights in the Markov chain. It is $D = abc - afg - bhi - cde + ife + dgh$.

$$T = \frac{\text{trace}(M)-1}{2}$$, is also dependent on the element of $P_1$.

As shown in Theorem 3.1, $D$ and $T$ influence the nature of the eigenvalues. Since the loop weights have an influence on both $D$ and $T$, we conjecture that loop weights and structure also determine spectra nature as shown below.
Corollary 3.18

From Theorem 3.1 and 3.2, the following nature of the D affect the nature of the eigenvalues (other than λ=1) of 3-state Markov chains as follow:

(i) If \( D = 0 \), then the eigenvalues are 1, 2T, 0. (Recall that \( T = \frac{a+b+c-1}{2} \))

(ii) If \( D < 0 \), then the eigenvalues (other than the leading 1) are real, one of the eigenvalues is negative and the other is positive.

(iii) If \( D > 0 \), then there are 3 possible eigenvalues nature based on its relationship with positive number, \( T^2 \).

1. If \( D = T^2 \), then \( \lambda = 1, T, T \) (double root).
2. If \( D > T^2 \), then \( \lambda = 1, T \pm xi \) (complex number).
3. If \( D < T^2 \), then \( \lambda = 1, T \pm x \) (two positive or two negative).

Proof

(i) Suppose \( D = 0 \).

By Theorem 3.1, \( \lambda = 1, T + \sqrt{T^2 - 0} \) and \( T - \sqrt{T^2 - 0} \).

\( \lambda = 1, T + T \) and \( T - T \).

Thus, \( \lambda = 1, 2T, 0 \).

(ii) and (iii) are restatements from Theorem 3.2.

This implies that the values of D and T can give information about spectra nature.

Thus, the period loop weight, or loop total weight, L, should influence eigenvalue nature.

Figure 3f indicates 6 regions, separated by a 3 lines/curves. The curves divide the regions based on the nature of the eigenvalue. Thus, the curves represent the set of points where bifurcation (change in the nature of the eigenvalues) can take place. The curve and regions are described and shown in Figure 3f below.
Figure 3f: Eigenvalue nature of 3-state Markov chain as D and T change. Here, y axis = D, det(P) and x axis = t, Trace(P). The curve is $D = (t-1)^2/2 = T^2$. There are 6 regions of eigenvalue nature in the figure, where $a, b > 0$.

The curves where the nature of the eigenvalues changes with changes in parameter might be referred to as bifurcation points. Therefore, we have 3 bifurcation points described below.

Case $T = 0$.

The eigenvalues have the same magnitude but opposite direction at this point. For complex eigenvalues, the nature of the real part of the eigenvalues changes between positive and negative. For real eigenvalues, the nature of the eigenvalues with higher magnitude changes.

Case $D = 0$.

One of the eigenvalues becomes zero at this point. The nature of the eigenvalues changes between alternating sign eigenvalue and same sign when the eigenvalues are real.

Case $D = T^2$.

The eigenvalues have double root, 1, T, T at this set of points. The nature of the eigenvalues changes between real and complex eigenvalues at this bifurcation point.

The regions are:
Region 1: the nature of the eigenvalue is \(-a \pm bi\). That is, eigenvalues (other than 1) are complex with a negative real part. This is the eigenvalue region where cycle 3 path is dominant. For instance, a directed cycle on three vertices (Figure 1d) has it eigenvalues, \(1, -0.5 \pm i\sqrt{0.75}\) at the edge of this region. Another example is the digraph whose stochastic matrix is

\[
\begin{bmatrix}
  0.1 & 0.9 - \alpha & \alpha \\
  0.25 & 0.2 & 0.55 \\
  0.4 & 0.45 & 0.15
\end{bmatrix}
\]

(3.6.1)

When \(\alpha = 0.2\), the stochastic matrix is

\[
\begin{bmatrix}
  0.1 & 0.7 & 0.2 \\
  0.25 & 0.2 & 0.55 \\
  0.4 & 0.45 & 0.15
\end{bmatrix}
\]

and the digraph is below.

Period 1 loop weight elements are \(\{0.1, 0.2, 0.15\}\)
Period 2 loop weight elements are \(\{0.2475, 0.08, 0.175\}\)
Period 3 loop weight elements are \(\{0.0225, 0.154\}\)
The eigenvalues are \(1, -0.275 + 0.19203i, -0.275 - 0.19203i\). The total loop weights are 0.45, 1.005 and 0.5295 respectively.

Region 2: the nature of the eigenvalue is \(a \pm bi\). The eigenvalues (other than 1) are complex with a positive real part. This is the eigenvalue region where cycle 3 path is dominant with cycle 1 path. For instance, the digraph whose stochastic matrix is

\[
\begin{bmatrix}
  0.4 & 0.5 & 0.1 \\
  0.05 & 0.45 & 0.5 \\
  \beta & 0.5 - \beta & 0.5
\end{bmatrix}
\]

(3.6.2)
With \( \beta = 0.4 \), so that \( M = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.45 & 0.5 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \)

Period 1 loop weight elements are \{0.4, 0.45, 0.5\}
Period 2 loop weight elements are \{0.05, 0.04, 0.025\}
Period 3 loop weight elements are \{0.0005, 0.1\}
The eigenvalues are 1, 0.175+0.33072i, 0.175-0.33072i.
The total loop weights are 1.35, 0.23 and 0.3015 respectively.

**Region 3:** The eigenvalues (other than 1) are both negative. This is the eigenvalue region where period 2 is dominant. For instance, the digraph in equation (3.6.1) with

\[ \alpha = 0.6, \text{ so that } M = \begin{bmatrix} 0.4 & 0.5 & 0.6 \\ 0.25 & 0.2 & 0.55 \\ 0.4 & 0.45 & 0.15 \end{bmatrix}. \]

Period 1 loop weight elements are \{0.1, 0.2, 0.15\}
Period 2 loop weight elements are \{0.2475, 0.24, 0.075\}
Period 3 loop weight elements are \{0.0675, 0.066\}
The eigenvalues are 1, -0.12293, -0.42707.
The total loop weights are 0.45, 1.125 and 0.4005 respectively.

**Region 4:** The eigenvalues (other than 1) are both positive. This is the eigenvalue region where period 1 is dominant. For instance, the matrix in equation (3.6.2).

With \( \beta = 1 \), \( M = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.45 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \) has eigenvalues 1, +0.2781, +0.0719.

Period 1 loop weight elements are \{0.4, 0.45, 0.5\}
Period 2 loop weight elements are \{0.2, 0.01, 0.025\}
Period 3 loop weight elements are \{0.002, 0.025\}
The eigenvalues are 1, 0.27808, 0.071922.
The total loop weights are 1.35, 0.47 and 0.081 respectively.

**Region 5 and 6:** The eigenvalues (other than 1) are both real but have opposite signs. That is, one is positive and the other is negative. In region 5, the negative eigenvalue has a higher magnitude than the positive one. But in region 6, the positive eigenvalue has a higher magnitude. For instance, the digraph whose stochastic matrix is
Period 1 loop weight elements are {0.6, 0.1, 0.2}
Period 2 loop weight elements are {0.56, 0.025, 0.015}
Period 3 loop weight elements are {0.0175, 0.012}
The eigenvalues are 0.5, 1, -0.6.
The total loop weights are 0.9, 1.2 and 0.0885 respectively.

If $\phi = 0.4$, then $M = \begin{bmatrix} 0.6 & 0.15 & 0.25 \\ 0.1 & 0.1 & 0.8 \\ 1 & 0.9 - \phi & \phi \end{bmatrix}$ with eigenvalues 1, 0.5, -0.4.

From the examples above, the total loop weight has an influence on the nature of the eigenvalues. The relationship will be explored below.

**Theorem 3.19: Eigenvalues and Periodicity** For any 3-state Markov chain eigenvalues (other than 1), the following are true.

(i) If $\lambda$ is real and $\lambda > 0$, then there exists one or more self-loops in the Markov chain. That is, Cycle 1 loop weight, $P_1$, has a non-zero element.

(ii) If $\lambda$ is real and $\lambda < 0$, then there exists one or more cycles of length 2 in the Markov chain. That is, Cycle 2 loop weight, $P_2$, has a non-zero element.

However, the converse of each statement is not always true.

**Proof**

(i) By way of contradiction, suppose there are no self-loops in a 3-state Markov chain. Then, $T = \frac{(0-1)}{2} = -0.5$ and $D = \sum P_3$. Thus, $D \geq 0$. By Theorem 3.1, the eigenvalues other than 1, are $-0.5 \pm \sqrt{0.25 - D}$. At least one of this number can only be real and positive if $0.25 - D > 0.25$. This implies that $D < 0$, which is a contradiction to $D \geq 0$. Therefore, there can be no real and positive eigenvalue if there is no self-loop or period 1 loop. Thus, there exist a real and positive eigenvalue
if there is a self-loop in the Markov chain.

Alternatively, suppose there is no self-loop in the Markov chain. Then, the sum of the three eigenvalues is 0, since the trace = sum of self-loops is 0.

Because, one of the eigenvalues is 1, the remaining two eigenvalues must sum to 1. Since the eigenvalues must be inside the unit circle in the complex plane, none of the two numbers can be real and positive.

(ii) By way of contradiction, suppose there is no period 2 loop in the Markov chain (that is $P_2 = 0$). We want to show that none of the eigenvalues is real and negative. Then, $D = abc + ife$ or $D = abc + dgh$ since $P_2 = 0$. Thus $D \geq 0$. By corollary 3.18, the following eigenvalues are possible when $D \geq 0$:

1. If $D = 0$, then $\lambda = 1, 2T, 0$
2. If $D = T^2$, then $\lambda = 1, T, T$ (double root)
3. If $D > T^2$, then $\lambda = 1, T \pm xi$ (complex number, $x \neq 0$)
4. If $D < T^2$, then $\lambda = 1, T \pm x$

It remains to show that none of these can be real and negative. It is obvious that (3) is not a real number.

There are two cases, reducible (not strongly connected) and irreducible (strongly connected) Markov chain.

Case 1: Reducible Markov chain

Suppose $M$ is reducible, then at least one of the self-loop weights, $a, b, \text{ or } c$, must be equal to 1. Thus, $\text{trace}(M) = a + b + c \geq 1$ which implies that $T = \frac{\text{trace} - 1}{2} \geq 0$.

Thus, (1) and (2) are not negative real numbers since $T \geq 0$.

Also, the determinant of a reducible $P$ is $D = abc$ or $D = 0$. 
Now observe that (4) can only have a negative real number if \( x > T \) (recall that \( x = \sqrt{T^2 - D} \)). However, \( x < T \) because \( D > 0 \).

Therefore, the eigenvalues’ nature in (4) cannot be a negative real number as well.

**Case 2:** Irreducible 3-state Markov chain with \( P_2 = 0 \).

Suppose \( M \) is irreducible and \( \sum P_2 = 0 \), then \( D = abc + (1 - a)(1 - b)(1 - c) \).

Thus,

\[
D = abc + (1 - (a + b + c) + (ab + bc + ac) - abc) \\
= 1 - (a + b + c) + (ab + bc + ac).
\]

Hence,

\[
D - 1 = (ab + bc + ac) - (a + b + c). \tag{3.3}
\]

Also,

\[
T^2 = \frac{(\text{trace}-1)^2}{4} \\
= a^2 + b^2 + c^2 + 2(ab + bc + ca) - 2(a + b + c) + 1 \\
= a^2 + b^2 + c^2 + 2(D - 1) + 1 \quad \text{by substituting equation (3.3)} \\
= a^2 + b^2 + c^2 + 2D - 1 \\
= \frac{D}{2} + \frac{a^2 + b^2 + c^2 - 1}{4}. \tag{3.4}
\]

Irreducible \( M \) can only have a negative real eigenvalue if \( T < 0 \) and \( D < T^2 \) (since \( D \geq 0 \), this is the only way for \( T \pm \sqrt{T^2 - D} \) to have a negative eigenvalue).

Suppose, \( T < 0 \). It remains to show that \( D > T^2 \) (or \( T^2 < D \)).

Observe that,

\[
a^2 + b^2 + c^2 < a + b + c \quad \text{since } a, b, c \text{ are } [0, 1]
\]
< 1 since T < 0 implies trace < 1.

Thus, \( a^2 + b^2 + c^2 < 1 \). This implies that \( \frac{a^2 + b^2 + c^2 - 1}{4} < 0 \).

Substituting equation (3.4). We have

\[
T^2 = \frac{D}{2} + \frac{a^2 + b^2 + c^2 - 1}{4}
\]

\[
< \frac{D}{2} \text{ Since } \frac{a^2 + b^2 + c^2 - 1}{4} < 0
\]

\[
< D.
\]

Thus, \( T^2 < D \). 

Therefore, a negative real eigenvalue indicates the presence of a period 2 loop \((P_2 \neq 0)\).

An extension of Theorem 3.19 to a cycle length of 2 is given below. However, there is no complete proof for it yet, so it is left as a conjecture in this dissertation.

**Conjecture 3.20** If \( \lambda \) is a complex eigenvalue, then there exist one or more cycles of length 3 in the Markov chain (and sample path). That is, cycle 3 loop weight, \( P_3 \), has a non-zero element.

**Proof:** (one case)

Case 1: Homoprobability

Observe that a cycle, for instance, the \( C_3 \) digraph in Figure 1d, has complex eigenvalues. Also, loop homoprobability chains have complex eigenvalues except when \( a+2b=1 \).

Suppose a 3-state Markov chain has complex eigenvalues, then \( D > T^2 \) by Theorem 3.2(iii). Now, \( D > T^2 \) implies that \( D \not< T^2 \). Thus, the statement of conjecture 3.20 can be restated as “if D is not less than or equal to \( T^2 \), then \( P_3 \neq 0 \).

Recall that a statement and its contrapositive are equivalent. Therefore, we will prove
the contrapositive of the statement in quote. The contrapositive is:

If \( P_3 = 0 \), then \( D \leq T^2 \).

Observe that \( P_3 = 0 \) in line homoprobability. Using line homoprobability, the conclusion of the statement becomes \( D \leq a^2(T = a \text{ for line homoprobability}) \). The determinant of line homoprobability is

\[
D = a(a + b)(1 - b) - (a + b)b(1 - a - b) - (1 - b)b(1 - a - b).
\]

Then, \( D \leq a(a + b)(1 - b) \) since a Markov chain has nonnegative numbers. Thus, it remains to show that \( a(a + b)(1 - b) \leq a^2 \).

Observe that \( b \leq 1 \)

\[
\leq 1 + a.
\]

Multiplying both sides by \( b \),

\[
b^2 \leq ab + b \\
0 \leq ab + b - b^2 \\
a \leq ab + b - b^2 + a \\
a^2 \leq a^2b + ab - ab^2 + a^2 \\
a^2 - a^2b - ab + ab^2 \leq a^2 \\
a(a - ab - b + b^2) \leq a^2 \\
a(1 - b)(a - b) \leq a^2.
\]

Thus,

\[
D \leq a(1 - b)(a - b) \leq a^2 \\
D \leq a^2.
\]

Note that the converse of each statement is not always true. For instance, consider the example below.
Example 3.21 In Figure 1g, the Markov chain has no self-loop. Hence, none of the eigenvalues apart from \( \lambda = 1 \) is positive. The remaining two eigenvalues are negative indicating period 2. However, there is no complex eigenvalue even though there is a period 3 loop in the Markov chain.

This results in the question, which of the loops or loop periodicity in the chain is picked up by the spectra? It was conjectured that there is always a dominant or skeleton loop in a Markov chain that determines a sample path to limit distribution and nature of an eigenvalue. However, we found a counterexample to this conjecture, which indicated that this is not always the case. For instance, the digraph in Figure 4a is a counterexample. The nature and value of the eigenvalues remain the same irrespective of the value of \( \alpha \). That is, the nature of the eigenvalue is determined by the weights of both period 2 loops.

The case where there is a unique loop that determines the eigenvalue nature is when only one of the loops in each periodic set is present. For instance, the figure below has only one of each period loop.

Thus, the nature of the eigenvalues indicates the sets of dominant loop period that describe the sample path to limit distribution. The contribution of each of the elements of this set to the spectral nature may differ significantly (as in Figure 4a). This results in the following questions. What are the dominant sets of spectral loop? And how can they be determined?
3.6.2 Dominant Set of Spectra Loops

**Definition 3.22:** The set of spectra loops are the set of loops whose structure is reflected by the nature of the eigenvalues.

The dominant set of spectral loops contains the loops whose periodic spectral representation has the higher magnitude. It describes the dominant cycle length of the sample path associated with the Markov chain. It can be determined by evaluating the nature of the eigenvalues.

Positive eigenvalues represent cycles on one vertex.

Negative eigenvalues represent cycles on two vertices.

Complex with negative real part represent cycle on 3 vertices.

Therefore, region 1 of Figure 3f is cycle length 3, region 2 is a blend of self-loops and cycle length 3, region 3 is cycle length 2, region 4 is cycle length 1 and regions 5 and 6 are a blend of self-loops and cycle length 2.

**Conjecture:** Given a stochastic digraph, there exists a dominant set of spectral loops, that completely determines the nature of the eigenvalues associated with the digraph. The following relationship was observed, however only the first one is proved mathematically in this dissertation.

**Conjecture 3.23:** If M is a 3-state Markov chain, the following loop structure relationship also determines the nature of the eigenvalue (other than \( \lambda = 1 \)).

(i) If one of the eigenvalues is a positive real number, then \( L_1 > 1 \).

(ii) If one of the eigenvalues is a negative real number, then \( L_2 > L_1 \).
(iii) If the eigenvalue is complex with an imaginary part, then $L_3 > \frac{L_2}{2}$.

A mathematical theorem of the interpretation of a positive real eigenvalue in a 3-state Markov chain is given below.

**Theorem 3.24** If $M$ is a 3-state Markov chain, the following loop structure determines the nature of the eigenvalue (other than $\lambda = 1$). The eigenvalues have a real and positive part if $L_1 > 1$.

**Proof**

(i) Recall that $L_1 = Trace(M)$ and $T = \frac{Trace(M) - 1}{2}$. Since $L_1 > 1$ then $T > 0$.

The eigenvalues (other than $\lambda = 1$), are $T \pm \sqrt{T^2 - D}$. There are 2 cases based on the relationship between $D = det(M)$ and $T^2$.

Case 1: $D > T^2$

If $D > T^2$, then $\lambda = T \pm \sqrt{T^2 - D}$ is $\lambda = T \pm xi$, complex conjugate with a positive real part.

Case 2: $D \leq T^2$

If $D \leq T^2$, then $\lambda = T \pm \sqrt{T^2 - D}$ is $\lambda = T \pm x$, where $x \geq 0$. At least, one of the real eigenvalues, $\lambda_2 = T + x$, must be a positive real number.

(Note that $\lambda_3 = T - x$ is a negative number if $x > T$ ($D < 0$) but $\lambda_2 = T + x$ will still have a higher magnitude.)

Thus, the eigenvalues (other than $\lambda = 1$) must have a positive real part.

Examples to support Conjecture 3.23 and Theorem 3.24 are given below.

**Example 3.25** Consider the Markov chains whose probability matrices are below.

(i) $M = \begin{bmatrix} .4 & .5 & .1 \\ .05 & .45 & .5 \\ .4 & .1 & .5 \end{bmatrix}$
The total loop weights are **1.35, 0.23 and 0.3015** respectively.

\[
L_1 > 1: 1.35 > 1? \quad \text{yes}
\]
\[
L_2 > L_1: 0.23 > 1.35? \quad \text{No}
\]
\[
L_3 > \frac{L_2}{2}: 0.3015 > 0.115? \quad \text{yes}
\]

Therefore, eigenvalue will have a complex conjugate with a positive real part. The actual eigenvalues are approximately \(1, \ 0.175+0.33072i, \ 0.175-0.33072i\).

(ii) \[
M = \begin{bmatrix}
0.1 & 0.3 & 0.6 \\
0.25 & 0.2 & 0.55 \\
0.4 & 0.45 & 0.15
\end{bmatrix}
\]

The total loop weights are **0.45, 1.125 and 0.4005** respectively.

\[
L_1 > 1: (0.45>1?) \quad \text{No}
\]
\[
L_2 > L_1: 1.125>0.45? \quad \text{Yes}
\]
\[
L_3 > \sum P_2 : 0.4005>0.5625? \quad \text{No}
\]

Therefore, there exist negative real eigenvalues in the spectrum. The actual eigenvalues are \(1, -0.12293, -0.42707\).

(iii) \[
M = \begin{bmatrix}
0.4 & 0.5 & 0.1 \\
0.05 & 0.45 & 0.5 \\
0.1 & 0.4 & 0.5
\end{bmatrix}
\]

The total loop weights are 1.35, 0.47 and 0.081 respectively.

\[
L_1 > 1: (1.35>1?) \quad \text{Yes}
\]
\[
L_2 > L_1: 0.47>1.35? \quad \text{No}
\]
\[
L_3 > \sum P_2 : 0.081>0.235? \quad \text{No}
\]
Therefore, there exist positive real eigenvalues in the spectrum. The actual eigenvalues are 1, 0.27808, 0.071922.

(iv) \( M = \begin{bmatrix} .6 & .15 & .25 \\ .1 & .1 & .8 \\ .1 & .7 & .2 \end{bmatrix} \)

The total loop weights are 0.9, 1.2 and 0.0885 respectively

\[
L_1 > 1: (0.9 > 1?) \quad \text{No}
\]
\[
L_2 > L_1: 1.2 > 0.9? \quad \text{yes}
\]
\[
L_3 > \sum P_2 : 0.0885 > 0.45? \quad \text{No}
\]

Therefore, there exist negative real eigenvalues in the spectrum. The actual eigenvalues are: 1, -0.6, 0.5

(v) \( M = \begin{bmatrix} .6 & .15 & .25 \\ .1 & .1 & .8 \\ .1 & .5 & .4 \end{bmatrix} \)

\[
L_1 = 1.1 > 1. \quad \text{Yes}
\]
\[
L_2 = 0.44 > L_1? \quad \text{No}
\]
\[
L_3 = 0.0245 > 0.22? \quad \text{No}
\]

Therefore, there exist positive real eigenvalues in the spectrum. The actual eigenvalues are \( \lambda = 1, 0.5, -0.4 \).

3.7 Coperiodic-Weight Markov Chain

We conjectured that Markov chains with similar spectral nature have some similarity in loop structure. This also implies that cospectral Markov chains have some similarity in loop structures.
Consider a Markov chain

\[
\begin{bmatrix}
a & d & i \\
e & b & g \\
h & f & c \\
\end{bmatrix}
\]

Let

- \(P_1\)-weight = \(\sum P_1 = a + b + c\) = (trace of the matrix)
- \(P_2\)-weight = \(\sum P_2 = fg + hi + de\)
- \(P_3\)-weight = \(\sum P_3 = dgh + ife\)

**Definition 3.26:** If two Markov chains have the same values of \(P_1\)-weight, \(P_2\)-weight, and \(P_3\)-weight, they are coperiodic.

For example, the two Markov chains below are coperiodic.

**Figure 3g** Examples of Coperiodic Markov chains

The first chain has only one period 2 loop and a period 3 loop. The second has 2 period 2 loops and a period 3 loop. However, they both have the same loop/period weight. For the first chain, \(P_1\)-weight is 0, \(P_2\)-weight 0.7 * 1 = 0.7 and \(P_3\)-weight is 0.3*1*1= 0.3. For the second chain, \(P_1\)-weight is 0, \(P_2\)-weight t is (0.5*1) + (0.4*0.5) =
0.5 + 0.2 = 0.7 and $P_3$-weight is $0.6 \times 0.5 \times 1 = 0.3$. Since, each of these 3 are the same, they are coperiodic.

We found counterexamples that showed cospectral chains are not necessarily coperiodic. For instance, the 2-state Markov chains with adjacency matrices, $\begin{bmatrix} 0.6 & 0.4 \\ 1 & 0.9 \end{bmatrix}$ and $\begin{bmatrix} 0.8 & 0.2 \\ 1 & 0.7 \end{bmatrix}$ are cospectral, but they are not coperiodic. However, the converse may be true, so that coperiodic set is a subset of cospectral. Although, we have not established that coperiodicity necessarily implies cospectral, we found a family of Markov chains where this is true.

### 3.7.1 A Family of Coperiodic Cospectral

The family of 3-state Markov chains with no self-loops has an interesting and unique property. The period weights always sums to 1. This implies that the period weight forms a probability distribution itself. This is not always the case for all Markov chains. Also, any digraphs in this family that are coperiodic are cospectral. The converse of this statement is also true for this family.

**Example 27** Consider the three adjacency matrices below:

$\begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0.5 & 0.5 & 0 \end{bmatrix}$ \hspace{1cm} $\begin{bmatrix} 0 & 1 & 0 \\ 0.1 & 0 & 0.9 \\ 0.3 & 0.7 & 0 \end{bmatrix}$ \hspace{1cm} $\begin{bmatrix} 0 & 0.27 & 0.73 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

These adjacency matrices have the same eigenvalues. That is, they are cospectral. Although, the structure of their corresponding digraphs seems different, they are coperiodic. Also, observe that the cycle loop weight all sum to 1.
Figure 3h Structure of the digraphs corresponding to the three adjacency matrices

For the first digraph, we have

\[ P_1 - \text{weight} = \{0, 0, 0\} \text{ and } \sum P_1 = 0, \]
\[ P_2 - \text{weight} = \{0.28, 0.3, 0.15\} \text{ and } \sum P_2 = 0.73, \]
\[ P_3 - \text{weight} = \{0.21, 0.06\} \text{ and } \sum P_3 = 0.27. \]

For the second Markov chain, we have,

\[ P_1 - \text{weight} = \{0, 0, 0\} \text{ and } \sum P_1 = 0, \]
\[ P_2 - \text{weight} = \{0, 0.1, 0.63\} \text{ and } \sum P_2 = 0.73, \]
\[ P_3 - \text{weight} = \{0, 0.27\} \text{ and } \sum P_3 = 0.27. \]

For the third Markov chain, we have

\[ P_1 - \text{weight} = \{0, 0, 0\} \text{ and } \sum P_1 = 0, \]
\[ P_2 - \text{weight} = \{0, 0, 0.73\} \text{ and } \sum P_2 = 0.73, \]
\[ P_3 - \text{weight} = \{0.27\} \text{ and } \sum P_3 = 0.27. \]

**Theorem 3:22** Suppose \( M_1, \) and \( M_2 \) are 3-state Markov chain, with no self-loops, then they are coperiodic, if and only if they are cospectral. (This is also applicable to any set \( M_1, M_2, \ldots, M_n \) for \( n \geq 2. \))
Proof

For this family of Markov chains, $T = -0.5$. Hence, $T^2 = 0.25$. Also, $D = P_3$-weight.

Suppose this is a coperiodic set, then the $P_3$-weight is the same for each digraph in the set. Hence, D is the same. Since D and $T^2$ are the same, the eigenvalues, $T \pm \sqrt{T^2 - D}$, are the same. Therefore, the set is cospectral.

Conversely, suppose the set of Markov chains is cospectral. Since, $T = -0.5$ for these set of Markov chains, $P_1$-weight is the same and the eigenvalues are $1, -0.5 \pm \sqrt{0.25 - D}$. This implies that D is the same across this set. Thus, $D = P_3$-weight is the same across the set. It remains to show that the $P_2$-weight is the same across the set.

Since the period weights for this family sum to 1,

\[ P_2\text{-weight} = 1 - P_3\text{-weight} - P_1\text{-weight} \]

\[ = 1 - D - 0. \]

This will be the same since D is the same.

Therefore, the set of digraphs is coperiodic. 

Chapter 4

BIFURCATION AND EIGENVALUES ATTRIBUTES

As introduced in Chapter 1.5, bifurcation is the change in the dynamics of a system with a changing parameter (Merrill, 2010). A bifurcation point is where the dynamics of the system changes. At this point, the sample path approach to limit distribution in a Markov chain changes. A change in the nature of the eigenvalues can indicates bifurcation, but the converse is not always so. For instance, the dynamic of the figure below changes as $\alpha$ becomes 0 but the eigenvalues remain the same.

![Figure 4a: Cospectra chain but different dynamics](image)

Thus, one context of bifurcation is the point when the nature of the eigenvalue changes as some parameter changes in the underlining chain. The nature of the eigenvalues changes from positive to negative (or vice versa) when one of the eigenvalues becomes zero. If we represent the chain in Figure 1e by $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ 1 - \alpha & \alpha \end{bmatrix}$ such that $\alpha = \frac{2}{3}$, the eigenvalues are both positive, $\lambda = 1, \frac{4}{15}$. When $\alpha$ decreases to $\frac{2}{5}$, the nature of the second eigenvalue is about to change to negative as the eigenvalues become $\lambda = 1, 0$.

The eigenvalue nature of Markov chains (for states bigger than 2 states) can also
change from real to complex (or vice versa) when the eigenvalues include a double root.

Suppose we have the transition matrix associated with the digraph below (the digraph is in Figure 4b and the probability matrix in equation 4.1). As the parameter $\alpha$ changes, the characteristic equation and, thus, the eigenvalues change.

$$
\begin{bmatrix}
0 & \alpha & 1 - \alpha \\
0.7 & 0 & 0.3 \\
0.55 & 0.45 & 0
\end{bmatrix}
$$ (4.1)

When $\alpha$ is 0.6, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -0.5 - 0.025 \approx -0.658$ and $\lambda_3 = -0.5 + 0.025 \approx -0.342$.

When $\alpha$ is reduced to 0.44, the eigenvalues become: $\lambda_1 = 1$, $\lambda_2 \approx -0.4684$, $\lambda_3 \approx -0.5316$.

When $\alpha = 0.43$, the eigenvalues are: $\lambda_1 = 1$, $\lambda_2 \approx -0.5 + 0.02236i$, and $\lambda_3 \approx -0.5 - 0.02236i$.

A bifurcation and change in the nature of the eigenvalues took place between $\alpha = 0.44$ and $\alpha = 0.43$. Specifically, the change took place when $\alpha = \frac{13}{30}$. At this value of $\alpha$, the characteristic equation of the transition matrix has a double root, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -0.5$.

Then, the nature of the eigenvalues changes to complex. Note that the multiplicity of $\lambda = 1$ represents the number of disconnected components.
In this Chapter, we will focus more on 3-state Markov chains since we have characterized their eigenvalues in Chapter 3.

4.1 Spectra Natures

In order to use the nature of eigenvalues to study the dynamics of a Markov chain, it is essential to understand the various natures that the eigenvalues can take on. Thus, we discuss the various natures of the eigenvalues of a general 3-state Markov chain. A 3-state Markov chain will always have 1 as one of its eigenvalues. The remaining two eigenvalues can be real or complex numbers whose magnitude is less than or equal to 1 (as indicated in Theorem 1.4.8). In this section, we focus on the nature of the remaining two eigenvalues.

4.1.1 Real Eigenvalues

When the eigenvalues of a Markov chain are real numbers, the following are possible.

1. \((++)\): This is when both eigenvalues are positive numbers with different magnitudes.
2. \((- -)\): This is when both eigenvalues are negative numbers with different magnitudes.
3. \((+ -)\): This is when one of the eigenvalues is positive and the other is negative. However, the positive eigenvalue has a higher magnitude.
4. \((- +)\): This is when one of the eigenvalues is positive and the other is negative. However, the negative eigenvalue has a higher magnitude.
5. \(\pm\): This is when one of the eigenvalues is positive and the other is negative. Also, they both have the same magnitude.

6. \(\pm\): this is when both eigenvalues are negative numbers with the same magnitudes. The double root has a component in the nature.

7. \(\pm\): this is when both eigenvalues are positive numbers with the same magnitudes. This is the other component of a double root.

8. \(0\): one or both of the eigenvalues are zero.

### 4.1.2 Complex Eigenvalues

When the eigenvalues of a 3-state Markov chain are complex numbers, the following are possible.

1. \((a\pm bi)\): This is a complex nature with a positive real part. However, the magnitude of \(a\) is greater than \(b\).

2. \((-a\pm bi)\): This is a complex nature with a negative real part. Also, the magnitude of \(a\) is greater than \(b\).

3. \((\pm bi-a)\): This is a complex nature with a negative real part. But, the magnitude of \(b\) is more than the magnitude of \(a\).

4. \((\pm bi+a)\): This is a complex nature with a positive real part. In addition, the magnitude of \(b\) is more than the magnitude of \(a\).

5. \((\pm bi)\): this is a purely imaginary nature. That is, the real part is 0.

6. \([a\pm bi]\): the points where \(a\) and \(b\) have the same magnitude.

7. \([-a\pm bi]\): the points where \(a\) and \(b\) have the same magnitude, with a positive real part.
4.2 Spectra Identification of Loop Structure

The nature of the eigenvalues can describe some of the loop structures in a 3-state Markov chain. As shown in Chapter 3.6, a positive eigenvalue indicates the presence of a self-loop (period 1), a negative eigenvalue indicates the presence of a period 2 loop (cycle paths of length 2) in a 3-state chain, and a complex eigenvalue indicates the presence of a period 3 loop (cycle paths of length 3). We also saw that a complex eigenvalue with a positive real part represents the presence of a self-loop in addition to a cycle path of length 3. However, a complex eigenvalue with a negative real part does not necessarily indicate the presence of a period 2 loop in addition to the period 3 loop. A cycle on 3 states is an example (Figure 1d). Its eigenvalues are $1, -0.5 + i\sqrt{0.75}, -0.5 - i\sqrt{0.75}$, but it has no period 2 loop in its chain path. The condition for each of these natures was discussed in the Chapter 3.6 regions and bifurcation curve. The natures $(a \pm bi)$ and $(\pm bi + a)$ were described together as nature $(a \pm bi)$. Similarly, natures $(-a \pm bi)$ and $(\pm bi - a)$ were described together as nature $(-a \pm bi)$.

Another spectral attribute of loop structure in Markov chains is when all or a portion of the eigenvalues are on a circle in the complex plane. The radius of the circle could be less than or equal to 1. However, if the radius of the circle is less than 1, the Markov chain is reducible and is usually applicable to a Markov chain whose states are greater than or equal to 3.

4.2.1 Eigenvalues on a circle in (or on) the unit circle in the complex plane

As shown in Proposition 1.3.5 and Lemma 1.3.7 in Chapter 1, the eigenvalues of the adjacency matrix of a weighted directed cycle satisfy $\lambda^n = \omega$. Here $\omega$ is the product
of the weights. The eigenvalues that satisfy this condition are equidistant from the origin in the complex plane. Hence, they are on a circle of radius $\sqrt[n]{\omega}$ for n-state Markov chain.

As observed in Example 1.4.13 with Figure 1f, an eigenvalue on a circle whose radius is less that 1 indicates a cycle on a transient component. Thus, the eigenvalues of Example 1.4.13 are on the unit circle and a circle of radius $\sqrt{0.7} \approx 0.8879$. An extension of this example is given below.

**Example 4.1** Consider, the Markov chain in Figure 4c. The chain has 3 strong components and each of these components is a directed cycle. The characteristic equation is

$$(\lambda^3 - 0.7) (\lambda^2 - 1) (\lambda^3 - 0.5) = 0.$$ 

The eigenvalues are the eigenvalues of each of the strong components.

Component 1: $\lambda \approx 0.8879, -0.444 + 0.769i, -0.444 - 0.769i$ (represented by * in Figure 4d)

Component 2: $\lambda = 1, -1$ (represented by “o” in Figure 4d)

Component 3: $\lambda \approx 0.7937, -0.3969 \pm 0.6874i$ (represented by “+” in Figure 4d)
Figure 4d: The eigenvalues of the digraph in Figure 4c on circles of different radii, 1, 0.888 and 0.7937 respectively.

4.3 Spectral Identification of Bifurcation

A bifurcation in a Markov chain might be identified by changes in the eigenvalues. Some of these changes could be changes in the nature of the eigenvalues. Other changes could maintain the same nature but change in magnitude. We will examine this in the following sections.

4.3.1 Nature Change

Changes between the eigenvalues’ natures stated in Chapter 4.1 might also indicate bifurcation. Some of the changes are major while others are minor. The changes are described below with their associated nature.

I. Eigenvalues Approach Zero.

At least one of the eigenvalues becomes 0 when the determinant is zero. That is, for a 3-state Markov chain, $\lambda = 1, 2T$ and 0 from Theorem 3.1. And if the trace is 1, both eigenvalues will be zero, since $T = 0$. The nature of the eigenvalues of this bifurcation class is (0). This will indicate a change in the dominant set of a spectral loop between
self-loops and period 2 loops.

II. Eigenvalues nature changes from real to complex or vice versa.

Eigenvalues’ change nature between real and complex numbers, when $D = T^2$. There is a double root in the spectrum at this point. The nature of the eigenvalues of this bifurcation class is (=) and (≠).

This indicates a change in the dominant set of spectral loop from period 3 loops or to period 3 loops (cycle paths of length 3). For instance, the Markov chain below has a bifurcation point at $\beta = 0.25$. Thus, the eigenvalues have double roots, $\lambda = 1, -0.5, -0.5$ at $\beta = 0.25$. When $\beta > 0.25$, the eigenvalues have a complex conjugate with a negative real part.

III. Same magnitude, but different direction

This change happens when trace is 1. The nature of the eigenvalues of this bifurcation class is (±) and (±bi).

This nature change could be between (±bi+a) and (±bi - a), (+) and (-) or even between (++) and (--).

IV. Complex eigenvalue with the same magnitude of real and complex parts.

Lastly, another nature change for complex eigenvalues is when the magnitude of the real
and imaginary part of the eigenvalues are the same. This happens when \( D = 2T^2 \). The nature of the eigenvalues of this bifurcation class is \([a \pm bi]\) and \([-a \pm bi]\).

### 4.3.2 Circles in the Complex Plane

Some changes in dynamic might not be accompanied by a change in the nature of an eigenvalue but a change in magnitude. For example, a change from

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
.7 & 0 & .3 \\
0 & 0 & 1
\end{bmatrix}
\]

did not change the nature of the eigenvalues but changed the magnitude from 1, 1,-1 to \(1, -\sqrt{0.7}, +\sqrt{0.7}\) (approximately 1, .8367, -.8367). This can be described as a change in dynamics because the digraph changed from a disconnected digraph to a connected digraph. Also, the spectral radius decreased.

Consider the adjacency matrix of the digraph in Figure 1f. The eigenvalues are approximately 1, \(-1, 0.8879, -0.44395 \pm 0.768947i\). Three of the eigenvalues are evenly distributed on a circle of radius \(\sqrt{0.7}\). The remaining 2 eigenvalues are evenly distributed on the unit cycle. The digraph becomes disconnected when the directed edge from state 2 to state 3 is deleted. The probability matrix changes from

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
.7 & 0 & 0 & .3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

The eigenvalues are now all on the unit circle. The eigenvalues becomes 1, \(-1, 1, -0.5 \pm \sqrt{0.75}i\). The maximum eigenvalue \(\lambda = 1\) now has a multiplicity of 2.
4.4 Sample Path Simulation and Sensitivity

We started the sample path simulation by using the generate function code (from Merrill 2010) in Appendix D. This function code takes a probability matrix \( P \), length \( n \) of the required number of sample path and an initial state where the simulated sample path begins from. Then, it generate another matrix, \( \text{cum} \), from \( P \) which is the cumulative sum of each row of matrix, \( P \). That is, the first column of \( \text{cum} \) is the same as the first column of \( P \) and the second column is the sum of the first and second column of \( P \). All other subsequent columns of \( \text{cum} \) are the sum of that column and the preceding columns in matrix \( P \). Since \( P \) is a transition matrix, the last column of matrix \( \text{cum} \) is always all ones.

The code generates the next state in the sample path by subtracting a random number from each element in the corresponding row state of \( \text{cum} \) and representing its signed values by a vector, \( \text{vec} \). If the product of the elements of \( \text{vec} \) is zero, the next state will be \( x(i + 1) = 0.5 \times (m - s) + 0.5 \), where \( s \) is the sum of vec and \( m \) is the length of matrix \( P \). If \( \text{prod(vec)} \) is not zero, state \( x(i + 1) = 0.5 \times (m - s) + 1 \). The sample path generated thus is the path of a single individual starting at the given initial state.

Let \( P = \begin{bmatrix} .2 & .8 & 0 \\ .3 & .4 & .3 \\ 0 & .4 & .6 \end{bmatrix} \). Then, we simulate a sample path of length 100, starting at state 1 as follows.

```matlab
x=generate(1,P,100)
pplot(x)
hold on
plot (x, 'x')
```
Figure 4f: Sample path generated from Matrix P by starting at state 1

\[ x = \]

Columns 91 through 100

2 1 1 2 3 3 2 2 2 1

Considering the last 10 states in the sample path, there are four loop 1 cycles, and no direct loop 2 or loop 3 cycles. Although, there were no direct loop 2 cycles, there were two indirect loop 2 cycles right after the self-loops. The last 10 sample paths did not show any indirect loop 3 cycles. Thus, the second largest eigenvalue is a positive real number which represents the self-loops. A loop 2 cycle is next, which is represented by a negative eigenvalue.

In comparison with the eigenvalues, the eigenvalues are \( 1, 0.1 \pm \sqrt{0.13} \approx 1, 0.4606, -0.2606 \). Observe that the second largest of these eigenvalues (in magnitude) is a positive real number and the lowest in magnitude is a negative real number. These eigenvalues, support the sample path observed from column 91 through 100 in the simulation.

The relationship between the sample path and the limit distribution can also be expressed by a well-known ergodic theorem, which states that as \( n \) tends to infinity, the
number of visits to state $i$ tends to $\frac{1}{m_i}$. Where $m_i$ is the expected return time to state $i$. For the matrix example above, the expected return time is approximately $\begin{bmatrix} 5.6689 \\ 2.1249 \\ 2.8337 \end{bmatrix}$. The mean (reciprocal) from the figure of the sample path above seems closely related to these values.

### 4.4.1 Sensitivity Analysis

The model is not sensitive to error except at or near a bifurcation point. That is, an error in a probability matrix may not affect the eigenvalue nature except the error is close to and crosses the bifurcation point. Therefore, the dynamics of a system is not sensitive to error except at the point where the dynamics changes. This results in the following question:

Given a matrix, how close is it to a bifurcation point? As discussed earlier in Chapter 3, there are three possible points where the nature of the eigenvalue can change. They are the point where $T = 0$, $D = 0$ and $D = T^2$. Recall that $D$ represents the determinant of a matrix $M$, $\text{det}(M)$, and $T = \frac{\text{Trace}(M) - 1}{2}$. Therefore, we will answer this question from the point of these 3 cases.

Case 1: $T = 0$. This is when the trace of the matrix is one. At this point, $\lambda_2$ and $\lambda_3$ have the same magnitude but opposite direction. Therefore, one measure of the closeness of a matrix to a bifurcation point is the closeness of its trace to 1.

Case 2: $D = 0$. This is when the determinant of the matrix is 0. At least, one of the eigenvalues is also 0 when $D=0$. Hence, another measure of closeness of a matrix to a bifurcation point, is the closeness of its determinant to 0.
Case 3: \( D = T^2 \). This is the point where a double root occurs in the eigenvalues. That is, \( \lambda_2 \) and \( \lambda_3 \) are the same. Thus, another measure of closeness of a matrix to a bifurcation point is when \( D \) is close to \( T^2 \).

Consider the matrix, 

\[
A = \begin{bmatrix}
0 & \alpha & 1 - \alpha \\
1 - \beta & 0 & \beta \\
\gamma & 1 - \gamma & 0
\end{bmatrix}
\]

Let \( \alpha = \frac{13}{30}, \beta = 0.3 \) and \( \gamma = \frac{55}{100} \). The determinant of \( A \), 

\[\det(A) = \alpha \beta \gamma + (1 - \alpha)(1 - \beta)(1 - \gamma) = 0.0715 + 0.1785 = 0.25.\]

Observe that \( T = \frac{0 - 1}{2} = 0.5 \). Hence, \( T^2 = 0.25 \). Therefore, matrix \( A \) is at a bifurcation case 3 because \( D = T^2 \). The eigenvalues are 1, \(-0.5\), \(-0.5\) (double root). At this point, the nature of the eigenvalues changes between real and complex numbers. If any of the parameters is reduced, \( \lambda_2 \) and \( \lambda_3 \) become complex conjugates. However, if any one of these parameters is increased, the eigenvalues are real numbers.

Therefore, given a probability matrix from data, the first step is to check the closeness of the matrix to a bifurcation point. If the matrix is close to a bifurcation point, a change in any of the parameter may change the nature of the eigenvalues depending on the direction of the parameter change (that is, if the parameter change crosses a bifurcation point).

**Example** Consider the matrix, 

\[
A = \begin{bmatrix}
0 & \alpha & 1 - \alpha \\
0.7 & 0 & 0.3 \\
0.55 & 0.45 & 0
\end{bmatrix}
\]

Let \( \alpha = 0.44. \) The eigenvalues of this matrix are 1, \(-0.5 - \sqrt{0.001}\) and \(-0.5 + \sqrt{0.001}\), which are both negative real numbers. Observe that for this matrix, \( D = 0.249 \) and \( T^2 = 0.25 \). Thus, matrix \( A \) is close to a case 3 bifurcation point. In terms of \( \alpha \), the eigenvalues are:

\[
\lambda = 1, -0.5 \pm \sqrt{0.15\alpha - 0.065} \tag{4.2}
\]
Now, suppose $\alpha$ is reduced slightly to $\alpha = 0.432$. Then, $D = 0.315 - 0.15\alpha = 0.2502$, crosses the bifurcation point and the discriminant, $T^2 - D = -0.0002$, is a negative number. The eigenvalues, $\lambda_2$ and $\lambda_3$ are $-0.5 \pm i\sqrt{0.0002}$ (approximately $-0.5 \pm 0.0141i$), which are complex conjugates. However, if $\alpha$ is increased slightly to $\alpha = 0.45$, $D = 0.315 - 0.15\alpha = 0.2475$. The bifurcation point is not crossed, and the matrix maintains the same eigenvalues nature ($\lambda_2 = -0.55$ and $\lambda_3 = -0.45$). This is also applicable to any other parameter that is changed in the matrix.

### 4.4.2 Sample Path Approach to Limit Distribution

Here, we want to observe the approach of the sample path to a limit distribution. We used a modification or update of the code in Merrill (2010), as shown in Appendix G, for a 2-state Markov chain and focused on the sample path approach to the limit distribution of state 0 (this is called $L_1$ in the code).

Theoretically, a matrix with a real positive eigenvalue has a monotonic approach to its limit distribution. Similarly, a Markov chain with a real negative eigenvalue has an oscillatory approach to its limit distribution. We illustrate these by using the following 2-state Markov chains.

Consider the 2-state Markov chain $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$. The eigenvalues are 1 and 0.7. Its limit distribution is approximately $\begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$. Hence, we used the code in Appendix G, to observe its monotonic sample path approach to the limit distribution of state 1 below.
Alternatively, the 2-state Markov chain, $P = \begin{bmatrix} 1 & 0.9 \\ 0.8 & 0.2 \end{bmatrix}$, whose eigenvalues are 1 and -0.7 approaches its limit distribution of approximately $\begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix}$ periodically. Although it approaches its limit distribution faster, the approach is oscillatory. The speed of approach to its limit distribution is as a result of starting at state 0, whose limit distribution may be far or close to the limit distribution of state 1.
Figure 4h: Oscillatory approach to limit distribution of state 1
Chapter 5

APPLICATION, DATA ANALYSIS AND RESULT

This Chapter discusses the application of the works introduced in this dissertation in corridor and metapopulation ecology.

5.1 Application to Habitat Fragment

Let each habitat fragment/patch be represented as a state in a Markov chain. Then, the weighted edges of the Markov chain are the probability of moving from one patch to another or the probability of a self-loop. These probabilities may be determined from the observed sample path of telemetry datasets of tracked, GPS-collared species, using the step function model or the continuous time Markov chain Model (Merrill, 2010, Fortin et al 2005, Hanks et al, 2015). The dispersal dataset, by incorporating species’ behavioral states, may give a better estimation as tested and concluded by Blazquez-Cabrera et al. (2016).

Suppose a patch is introduced between two disconnected habitat patches, it may serve as a corridor, a habitat extension, both or neither. The Markov chain will be like the model in Figure 3e (which is also in Figure 5a (i) below). Hence, we find a mathematical quantity to evaluate the effectiveness of the introduced patch in facilitating movement of individuals between the fragment patches. We start by modeling the system as a Markov chain model with probability matrix, \( P = \begin{bmatrix} a & d & 0 \\ e & b & g \\ 0 & f & c \end{bmatrix} \). Then, measures from the probability matrix of the Markov chain will be used to evaluate the effectiveness of the added patch.
5.1.1 Eigenvalue and First Passage Time Measure of Effectiveness

In Chapter 3.5, we discussed the total probability of first return, called the *loop total weight*. They were subdivided into loop1 total weight, $L_1$, loop 2 total weight, $L_2$, and loop 3 total weight, $L_3$. Also, in Chapter 3.6, the relationship between the set of elements in each period weight and the corresponding total loop weight was shown.

When the fragmented patches are modeled as a Markov chain, the eigenvalues of the probability matrix can be computed. These will describe the structure of the dominant loop in the model. If one of the eigenvalues is negative, that implies loop 2 cycle movement. That is, there is an exchange of individuals between the introduced patch and at least one of the patches.

Also, the mean first passage time between patch 1 to patch 3 through the corridor (which is represented as patch 2) can be computed. These passage times were proved in Chapter 3 to be:

\[
M_{13} = \frac{1 + p_{12} - p_{22}}{p_{12} \cdot p_{23}} \quad \text{and} \quad M_{31} = \frac{1 + p_{32} - p_{22}}{p_{32} \cdot p_{21}}.
\]

These passage time measures together with the eigenvalues can be used to evaluate the effectiveness of the introduced habitat patch as a corridor or habitat enlargement. We define the passage time, PT, as $M_{13} + M_{31}$. Consider Figure 5a below.
The adjacency matrix to the digraph in Figure 5a(i) is
\[
\begin{bmatrix}
p_{11} & p_{12} & 0 \\
p_{21} & p_{22} & p_{23} \\
0 & p_{32} & p_{33}
\end{bmatrix}
= \begin{bmatrix}
a & d & 0 \\
e & b & g \\
0 & f & c
\end{bmatrix}.
\]

The adjacency matrix to the digraph in Figure 5a(ii) is
\[
\begin{bmatrix}
p_{11} & p_{12} & 0 \\
p_{21} & p_{22} & p_{23} \\
0 & p_{32} & p_{33}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
1 - \alpha & 0 & \alpha \\
0 & 1 & 0
\end{bmatrix}.
\]

The Markov chain in Figure 5a(ii) has only period 2 loops in it. The eigenvalues are 1, -1, and 0, irrespective of the value of \( \alpha \). The negative eigenvalue indicates the facilitation of movement between at least one pair of states. However, the passage time,

\[PT = \frac{1 + 1 - 0}{\alpha} + \frac{1 + 1 - 0}{(1 - \alpha)} = \frac{2}{\alpha(1 - \alpha)},\]

changes significantly with \( \alpha \). PT tends toward infinity as \( \alpha \) tends toward 0 or 1. This indicates the necessity, but insufficiency of the eigenvalue as a measure for effectiveness. The minimum value of PT is 8. And it occurs when \( \alpha = 0.5 \).

This is also the point where \( \alpha = 1 - \alpha \), and \( \text{abs}(M_{13} - M_{31}) = 0 \), indicating that these might play a role in measuring the effectiveness.

\[
\lim_{\alpha \to 0^+} (PT) = \infty
\]

\[
\lim_{\alpha \to 1^-} (PT) = \infty
\]
\[
\frac{d(PT)}{d\alpha} = \frac{2(2\alpha - 1)}{\alpha^2(1-\alpha)^2} = \frac{4\alpha - 2}{\alpha^2 - 2\alpha^3 + \alpha^4}
\]

If \( b \neq 0 \), the total passage time will be \( PT = \frac{2 - 3b - b^2}{\alpha(1 - \alpha - b)} \). PT changes but still maintains the same shapes as shown below.

5.1.2 Effectiveness ID: Habitat Extension, Corridor, Both or None

We conjecture that two conditions are necessary for a patch to be effective as a corridor that facilitates the movement of individuals between two fragmented habitat patches.
1. Negative eigenvalue: A negative eigenvalue indicates the presence of a cycle 2 loop in the Markov chain model. This means that the introduced patch is either effective as a patch enlargement for one of the patches or as a corridor to facilitate movement between the two patches.

2. Small PD, where \( PD = \frac{|M_{13} - M_{31}|}{\min(M_{13}, M_{31})} \). That is, PD < 1. This ensures that both cycle 2 loops in the model contributed to the cycle 2 eigenvalue nature. Hence, the introduced patch is serving as a corridor that facilitates movement of species across disconnected habitat fragments. This condition will not be needed if condition 1 fails.

Although it seems reasonable to expect that it will be necessary for the total first passage time to be small, this has been taken care of by the two conditions above.

![Figure 5d: Eigenvalues vs Passage Time total. It increases as the eigenvalues become bigger.](image)

Consider the Markov chain model below.

**Example** This example uses a time-driven model for the probability matrix. That is, in a given time step, \( \Delta t \), a species from \( S_0 \) cannot reach \( S_2 \) and vice versa.
Figure 5e: Markov chain model of corridor and habitat patches

\[
P = \begin{bmatrix}
1 - p_1 & p_1 & 0 \\
p_4 & 1 - (p_3 + p_4) & p_3 \\
0 & p_2 & 1 - p_2
\end{bmatrix}
\]

If the limit distribution is set to be \(X = [\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}]\). Then solving the equation \(P'X' = X'\) for \(p_3\) and \(p_4\), the transition matrix becomes

\[
P = \begin{bmatrix}
1 - p_1 & p_1 & 0 \\
0.5p_1 & 1 - 0.5(p_1 + p_2) & 0.5p_2 \\
0 & p_2 & 1 - p_2
\end{bmatrix}
\]

- When \(p_1 = 0.1\) and \(p_2 = 0.2\)

\[
\lambda = 1, 0.8781, 0.6719
\]

\[
PT = 25 + 35 = 60
\]

\[
PD = 10/25 = 0.4.
\]

The positive eigenvalues indicate that the self-loops are still dominating. Hence, the patch is not very effective as a habitat extension or corridor. As expected from the positive eigenvalues, passage time, \(PT\), is high (relative to 2 and 4 which are the first return times). PD is irrelevant here since condition 1 failed.

- When \(p_1 = 0.1\) and \(p_2 = 0.9\)

\[
\lambda = 1, 0.8685, -0.3685,
\]
PT = 13.3 + 31.1 = 44.4,

PD = 17.8/13.3 = 1.3383.

One of the eigenvalues is negative, which indicates the effectiveness of the corridor. However, PD > 1 which indicates that the effectiveness serves more as a habitat extension to one of the patches.

- When $p_1 = 0.5$ and $p_2 = 0.5$
  
  \[ \lambda = 1, 0.5, 0, \]
  
  \[ PT = 8 + 8 = 16, \]
  
  \[ PD = 0. \]

This is indicating a bifurcation point between cycle one and two in the sample path.

- When $p_1 = 0.6$ and $p_2 = 0.55$
  
  \[ \lambda = 1, 0.4272, -0.1522 \]
  
  \[ PT = 7.12 + 6.81 = 13.94 \]
  
  \[ PD = 0.31/6.81 = 0.0455. \]

The negative eigenvalue indicates the effectiveness of the introduced patch. The fact that PD < 1 and is very close to 0 is an indication that the introduced patch is a corridor and effective in facilitating transition between the two fragmented patches.

- When $p_1 = 0.8$ and $p_2 = 0.7$
  
  \[ \lambda = 1, -0.5066, 0.2566, \]
  
  \[ PT = 5.536 + 5.179 = 10.71, \]
  
  \[ PD = 0.357/5.536 = 0.0645. \]
Similarly, the negative eigenvalue indicates the effectiveness of the introduced patch and PD << 1 is an indication that the introduced patch is a corridor and effective in facilitating transition between the two fragmented patches.

In conclusion, an introduced habitat patch between two fragmented landscapes is effective as a corridor and facilitates movement if it has a negative eigenvalue and PD<1.

5.2. Test and Discussion of Real Corridors.

In this section, we use information gathered from Dixon et al (2006) to test the hypothesis. We introduce the corridor, state the method used, give and analyze results. We concluded with a discussion and some limitations of the test.

5.2.1. Introduction to Osceola - Ocala Black Bear Habitat

Dixon et al (2006) analyzed 112 black bears whose captured hair were examined via population assignment. Seventy of the black bears were from Ocala, 39 from Osceola, 2 were mixed and one had an inconclusive assignment. Of the 70 Ocala bears, 40 were found in Ocala, 28 in the corridor and 2 in Osceola. The distribution is as follows.

Ocala: 40 black bears were found in Ocala and they all originated from Ocala.

Corridor: 31 bears were noted in the corridor. Twenty-eight of them are from Ocala and 3 from Osceola.

Osceola: 41 bears were noted in Osceola. Thirty-six of them are from Osceola, two from Ocala, one was inconclusive and the remaining 2 have mixed Osceola-Ocala origin.
5.2.2 Methods and Data

We are assuming that the process is already at limit distribution. Suppose \( X \) is the limit distribution and the probability matrix \( P \) is

\[
\begin{bmatrix}
1 - p_1 & p_1 & 0 \\
p_4 & 1 - p_4 - p_3 & p_3 \\
0 & p_2 & 1 - p_2
\end{bmatrix}
\]

We solve the equations from \( P'X' = X' \) for \( p_3 \) and \( p_4 \) in terms of \( p_1 \) and \( p_2 \). Thus, \( P \) is in terms of two unknowns, \( p_1 \) and \( p_2 \). We use three different limit distributions in three cases below.

\[\text{Figure 5f: Markov chain model of Osceola-Ocala Corridor and habitat patch}\]

Case 1

Let the limit distribution be the black bear distribution of the 111 bears, \( [\frac{40}{111} \quad \frac{31}{111} \quad \frac{40}{111}] \).

Using this limit distribution to solve for \( p_3 \) and \( p_4 \), the matrix becomes,

\[
\begin{bmatrix}
1 - p_1 & \frac{40}{31}p_1 & 0 \\
\frac{40}{31}p_1 & 1 - \frac{40}{31}(p_1 + p_2) & \frac{40}{31}p_2 \\
0 & p_2 & 1 - p_2
\end{bmatrix}
\]

Case 2

If the Ocala distribution of the 70 black bears is used as limit distribution,

\[
X = \begin{bmatrix}
\frac{40}{70} & \frac{28}{70} & \frac{2}{70}
\end{bmatrix}
\]
The stochastic matrix becomes
\[
\begin{bmatrix}
1 - p_1 & p_1 & 0 \\
\frac{10}{7} p_1 & 1 - \frac{10}{7} p_1 - \frac{1}{14} p_2 & \frac{1}{14} p_2 \\
0 & p_2 & 1 - p_2
\end{bmatrix}.
\]

Case 3

If the Osceola distribution of the 39 black bears is used as limit distribution,
\[
X = \begin{bmatrix} 0 & 3 & 36 \\ 39 & 39 & 39 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{13} & \frac{12}{13} \end{bmatrix}.
\]

The stochastic matrix becomes
\[
\begin{bmatrix}
1 - p_1 & p_1 & 0 \\
0 & 1 - 12p_2 & 12p_2 \\
0 & p_2 & 1 - p_2
\end{bmatrix}.
\]

Each of these matrices now has two unknowns, \(p_1\) and \(p_2\).

Next, we need the probability that a species will leave Ocala, \(p_1\), and the probability a species will leave Osceola, \(p_2\). At first, we assume that \(p_1\) is the fraction of Ocala bears that are not found in Ocala and \(p_2\) is the fraction of Osceola bears that are found in Osceola. That is, \(p_1=3/7\) and \(p_2=1/13\). We also did a simulation of this sample path as it approaches limit distribution. The results of this assumption are in Result case 1 to 3 in Chapter 5.2.3 below.

Dixon et al (2006) also observed that three of the Ocala individuals were spotted first in Ocala and eventually also spotted in the corridor. Hence, we may also assume that
the probability of leaving Ocala is, \( p_1 = \frac{3}{40} \). In addition, one of the three Osceola bears found in the corridor, was initially spotted in Oseola (North of I-10). This was why it was concluded that I-10 was not a complete barrier but a filter to the movement of the black bears. Thus, we also assumed that the probability of leaving Osceola in one time step is, 
\( p_2 = \frac{1}{41} \). However, this will not be a good measure of probability of movement.

5.2.3 Results and Analysis

Case 1:

\[
P = \begin{pmatrix}
1 - p_1 & p_1 & 0 \\
(40/31) * p_1 & 1 - (40/31) * p_1 - (41/31) * p_2 & (41/31) * p_2 \\
0 & p_2 & 1 - p_2
\end{pmatrix};
\]

Period 1 loop weight elements are \{0.57143, 0.3475, 0.92308\}
Period 2 loop weight elements are \{0.007635, 0, 0.237\}
Period 3 loop weight elements are \{0, 0\}
The eigenvalues are \{-0.044759, 1, 0.88701\}.
The total loop weights are 1.8398, 0.48965 and 0 respectively.
The passage times are 25.4083, 17.1417 and totals 42.55.
The PD is 0.48226.

\[
x = \text{generate}(1, P, 100)
\]

\[
x = 
\begin{pmatrix}
3 & 2 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\
2
\end{pmatrix}
\]

By observing the last ten sample paths, there are four self loops (loop 1 cycles).
There is also one loop 2 cycle and no loop 3 cycle. This was also reinforced by the computed eigenvalues.
Although there is a negative eigenvalue in case 1 above, the magnitude is relatively small. It is the eigenvalue with the smallest magnitude. This implies that as the sample approaches limit distribution, loop 2 cycles may occur, but are fewer. However, the positive eigenvalue has a much higher magnitude. This implies that the bears are staying and not leaving the initial habitat. Although, the PD is less than 1, this criteria is not as strong as the eigenvalue criteria. The eigenvalue criteria is more dominant. Also, the low value of PD is more likely due to the use of the overall bear distribution as a limit distribution. The black bear distribution does not portray a bear’s ability to move around.

Case 2

\[ P = [1-p1 \ p1 \ 0; (10/7)^*p1 \ 1-(10/7)^*p1-(1/14)^*p2 \ (1/14)^*p2;0 \ p2 \ 1-p2]; \]

Period 1 loop weight elements are \{0.57143, 0.38226, 0.92308\}
Period 2 loop weight elements are \{0.00042265, 0, 0.26239\}
Period 3 loop weight elements are \{0, 0\}
The eigenvalues are -0.044314, 1, 0.92108.
The total loop weights are 1.8768, 0.52563 and 0 respectively.
The passage times are 444.3333, 14.75 and totals 459.0833.
The PD is 29.1243.

Columns 91 through 100

\[
\begin{array}{cccccccccc}
2 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

By observing the last ten sample paths, there are five self loops (loop 1 cycles). There is also one loop 2 cycle and no loop 3 cycle. This was also reinforced by the computed eigenvalues.

Case 2 has a negative eigenvalue, but its magnitude is relatively small and close to zero. Also, its PD is very high relative to 1. This indicates that the corridor does not
facilitate the movement of Ocala black bears back and forth as much as case 1. The negative eigenvalues describes the corridor as a habitat extension of Ocala black bears.

Case 3

\[ P = [1-p_1 \ p_1 \ 0;\ 0 \ 1-(12)*p_2 \ (12)*p_2;\ 0 \ p_2 \ 1-p_2] \]

Period 1 loop weight elements are \((0.57143, 0.076923, 0.92308)\)
Period 2 loop weight elements are \((0.071006, 0, 0)\)
Period 3 loop weight elements are \((0, 0)\)

The eigenvalues are 0.57143, 0, 1.
The total loop weights are 1.5714, 0.14201 and 0 respectively.
The passage times are 3.4167, Inf and totals Inf.
The PD is Inf.

Columns 91 through 10

\[
\begin{array}{cccccc}
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

There is no loop 2 cycle nor loop 3 cycle as the sample path approaches limit distribution.

This case only has positive eigenvalues. The infinity in PD also indicates that there is no mixing of individuals between the patches.

5.2.4 Discussion and Limitations

The result of this test indicated that the O2O corridor is not very effective in facilitating the exchange of black bears between the Osceola and Ocala habitats. The O2O corridor seems to act more as a habitat extension of the Ocala habitat. This could be a result of high density/overpopulation in Ocala as discussed by Dixon et al (2006). After the prohibition of bear hunting in 1971, the bear population in Ocala grew (Dixon et al,
Hence, Osceola bears, or even Ocala bears, may not want to stay or return to Ocala due to overpopulation. This also reinforces why the Ocala bears are moving into the corridor.

Although, the PD in case 1 was small, using the bear distribution as a limit distribution does not seem like an effective way of measuring the mixing of species. The Ocala distribution in case 2 seems to be the best limit distribution for evaluating the exchange of individuals through the corridor.

The Osceola distribution in case 3 has positive and zero eigenvalues. In fact, the eigenvalues are $1, 1 - p_1, 0$. This is because the Markov chain model is reducible and by one of the observations in Chapter 3.4, $1 - p_1$ will be one of the eigenvalues. Also, the probabilities of staying in the corridor or in Osceola sum to 1, which is responsible for the zero eigenvalue.

For the corridor to be more effective, we will assume that the distribution in case 2 is the limit distribution because it represents the spreading of Ocala bears into all the patches.

Although the data used for $p_1$ and $p_2$ may seem to be a limitation, there is not enough data to accurately determine these values. When we tried the other alternative discussed in the method section (5.2.2), $p_1 = \frac{3}{40}$ and $p_2 = \frac{1}{41}$, all the eigenvalues came out positive. This is obviously not a good measure. A good measure of the probability will be an extension of the approach of Clark et al (2015). The authors created 40 replicate projections of 100 randomly selected locations in each patch (4000 paths). The
fraction that intersected an adjacent patch was used to describe the connectivity between the patches. This is pretty similar to the probability of reaching a target area in a given time which is called the “transition probability” by Tischendorf and Wissel (1997).

The result and example in Chapter 4.4.1 also shows that the choice of these probabilities may have no effect except the probability matrix is close to a bifurcation point and a sensitive parameter is moved in the direction that crosses the bifurcation point. In addition, the test did not evaluate why the Osceola bears in the corridor are not entering Ocala. It could be a result of overpopulation in Ocala. Also, the data is insufficient to tell if Ocala bears in the corridor are returning to Ocala, considering the high population density in Ocala.

Other work similar to this approach was done by Green et al. (2018). The authors used camera traps to evaluate the use of the MKEC (Mount Kenya Elephant Corridor) as either a transit (corridor) or a habitat extension. They concluded that different parts of the corridor were used differently. The use is dependent on vegetation cover (corridor characteristics) or the level of human disturbance (Green et al. 2018).

Other works have focused on identifying habitat cores and corridors for certain species. For instance, Almansien et al. (2016) made an effort to give a detailed representation of the possible spatial distribution of the critically endangered Iranian Black Bears (IBB). This was necessary in order to identify the most important conservation areas for the species’ survival. The authors identified 31 habitat cores for IBB by using 270 presence points. The authors also used least-cost modelling
(Adriaensen et al., 2003) to identify about 55 possible habitat corridors in the Iran Bear distribution area (Almansien et al., 2016).

Blazquez-Cabrera et al. (2016) showed that the kind of data used to characterize corridors and effective distance can affect the connectivity estimate. Although data from dispersal movement are more expensive, they provide a better and more accurate estimate (Blazquez-Cabrera et al., 2016).

5.3 Metapopulation Application

The family of coperiodic cospectral digraphs (in Chapter 3.7) can be a very useful tool in metapopulation ecology. The landscape matrix, $M$, which is used to estimate metapopulation capacity can be transformed into this family by dividing each row by its row weight. In fact, the colonization matrix described in Ovaskeinan and Hanski (2003), is in this family if only 3 patches are considered. The limit distribution of the colonization matrix represents the contribution of each patch to colonization in the metapopulation (patch value).

For a 3 patch metapopulation model, only the maximum eigenvalue, $\lambda=1$ can be positive. None of the remaining 2 eigenvalues can be a positive real number by Theorem 3.19 i. In addition, the eigenvalues can only be complex numbers if $P_3 - \text{weight}, \sum P_3 > 0.25$. This is because it determinant, $D = \sum P_3$ and $T = -0.5$. Thus $D > T^2$ if $\sum P_3 > 0.25$. Therefore, we conjecture that a 3-patch metapopulation model of colonization matrix $B$ is more effective when, $\det(B) = D > 0.25$. 
CONCLUSION, RECOMMENDATION AND FUTURE WORK

In summary, the nature of the eigenvalues can give some information about the structure of a dynamic system. Similarly, the loop or cycle structure path of a Markov chain system is related to the nature of the eigenvalues. We found some necessary conditions on relating loop structure and eigenvalue nature. Also, comparison of the loop weights seems to determine this relationship sufficiently.

In order to evaluate the effectiveness of a corridor between two fragmented patches, each patch and corridor will be represented by a state in a Markov chain model. Time series data of tracked animal movement can be used to form the probability matrix as modeled by Merrill (2010). The introduced corridor area patch between two disconnected habitat patches is effective as a patch enlargement or a corridor if eigenvalues are negative. The patch is effective as a corridor if PD<1 in addition to a negative eigenvalue.

The next question is, if a patch is not effective, how can it be made effective? How can a corridor be made optimal? Blazquez-Cabrera et al. (2018) determined the optimal corridor whose restoration would enhance metapopulation connectivity of the Iberian Lynx. The authors compared current effective distance with a scenario of optimal effective distance. The connectivity gain was mainly affected by the length of the least-cost path and it decreases as species’ dispersal distance increases (Blazquez-Cabrera et al., 2018). A similar approach could be applied in our methodology.
6.1 Dissertation Contribution

This dissertation completely characterized the eigenvalues of any 3-state Markov chain. The eigenvalues of any 3-state Markov chain were described in terms of the determinant and trace of its associated probability matrix. It also gave mathematical proofs to some relationships between eigenvalue nature and loop structure identified in Merrill (2010). The eigenvalues of any weighted cycle (or a cycle that is a strong component of a weighted digraph) was also a contribution as an extension. A relationship between limit distribution and period 2 loops of a class of 3-state Markov chains was contributed.

The homoproability Markov chain was introduced and its eigenvalues was characterized. The eigenvalues of its loop form (for 3-state Markov chain) were also shown and proved to be on or inside the equilateral triangle whose vertices are on the cube roots of unity. The line homoproability Markov chain was used to prove the necessity of a period 3 loop for complex eigenvalues in a 3-state chain.

The loop structures and eigenvalues of a Markov chain were analyzed. Period weight and loop total weight were defined and related. The relationship between the loop total weights was used to completely relate eigenvalue nature and loop structure in conjecture 3.23. However, only the case when \( L_1 > 1 \) was given a complete mathematical proof. It was used to evaluate the effectiveness of an ecological corridor in connecting fragmented habitat patches in addition to a measure from the passage time.

This dissertation also introduced and defined a coperiodic weight Markov chain. A family of coperiodic cospectral digraphs was also identified and introduced. If this family and its properties is further explored, it will be a very useful tool in metapopulation ecology.
Most of the work in this dissertation is for 3-state Markov chains, we can extend this to states bigger than 3-states in the future. Also, the focus of this dissertation has been on Markov chains. The work can be extended to other types of weighted digraphs. One other digraph of particular interest is the competition chain of a digraph.

Cohen (1968) introduced and used the competition graph to analyze food webs in 1968. The concept was extended to the weighted competition graph (Sano, 2007). The concept was further extended by Faronbi and Factor (Unpublished manuscript, 2011) to the percentage-weighted competition graph. This is the competition chain with an arc weight represented as decimal number. It is similar to the Markov chain in the sense that the weight of the arcs is less than or equal to 1. However, there are no self-loops and the out-degrees of each node do not sum to 1. The eigenvalues and bifurcation measure could be applied to the competition chain to determine the survivability and susceptibility of species in an ecosystem.

The dominant eigenvalues may also have relationship with the work that has been done on domination graphs and it extensions (Fisher et al, 1988, Factor, & Langley, 1988). We intend to investigate and find possible relationships (if that exist) in the future.

The coperiodic weight Markov chain and the family of coperiodic cospectral introduced in this dissertation may have a very important application in metapopulation colonization if further explored. We intend to further develop this concept and explore how it can be used to measure metapopulation strength.
REFERENCES


Appendices

Appendix A
% Matlab Code for Regions in Chapter 3.6
clear all
close all
% input matrix P below
    P=[.6 .15 .25;.1 .1 .8;.1 .5 .4];
% P=[.4 .5 .1;.05 .45 .5;.1 .4 .5];
a=P(1,1);
b=P(2,2);
c=P(3,3);
d=P(1,2);
e=P(2,1);
f=P(3,2);
g=P(2,3);
h=P(3,1);
i=P(1,3);
X=['Period 1 loop weight elements are {' ,num2str(a),' , num2str(b),' , num2str(c),' } ];
disp(X)
X=['Period 2 loop weight elements are {' ,num2str(f*g),' , num2str(h*i),' , num2str(d*e),' } ];
disp(X)
X=['Period 3 loop weight elements are {' ,num2str(i*f*e),' , num2str(d*g*h),' } ];
disp(X)
E=eig(P);
Y=['The eigenvalues are ' ,num2str(E(1)),' , num2str(E(2)),' , num2str(E(3)),' ];
disp(Y)
L1 = a+b+c;
L2 = 2*((f*g)+(h*i)+(d*e));
L3 = 3*((d*g*h)+(i*f*e));
Z=['The total loop weights are ' ,num2str(L1),' , num2str(L2),' and ' ,num2str(L3),' respectively. ' ];
disp(Z)

Appendix B
% Code for Chapter 5
% O2O code with the bear distribution as Limit distribution
p1 = 3/7;
p2 = 1/13;
%P = [1 - p1 p1 0; 0.5*p1 1-0.5*(p1+p2) 0.5*p2; 0 p2 1- p2];
%P = [1-p1 p1 0; (10/7)*p1 1-(10/7)*p1-(1/14)*p2 (1/14)*p2; 0 p2 1-p2];
%P = [1 - p1 p1 0; (40/31)*p1 1-(40/31)*p1-(41/31)*p2 (41/31)*p2;0 p2 1-p2];
P = [1-p1 p1 0; 0 1-(12)*p2 (12)*p2;0 p2 1-p2]; % OS distribution

a=P(1,1);
b=P(2,2);
c=P(3,3);
d=P(1,2);
e=P(2,1);
f=P(3,2);
g=P(2,3);
h=P(3,1);
i=P(1,3);
X=['Period 1 loop weight elements are {',num2str(a),',',num2str(b),',',num2str(c)},'];
disp(X)
X=['Period 2 loop weight elements are {',num2str(f*g),',',num2str(h*i),',',num2str(d*e)},'];
disp(X)
X=['Period 3 loop weight elements are {',num2str(i*f*e),',',num2str(d*g*h)},'];
disp(X)
E=eig(P);
Y=['The eigenvalues are ',num2str(E(1)),',',num2str(E(2)),',',num2str(E(3)),'.'];
disp(Y)

L1 = a+b+c;
L2 = 2*((f*g)+(h*i)+(d*e));
L3 = 3*((d*g*h)+(i*f*e));
Z=['The total loop weights are ',num2str(L1),',',num2str(L2),' and ',num2str(L3),' respectively.'];
disp(Z)

M13= (1+ P(1,2) - P(2,2))/(P(1,2) * P(2,3));
M31= (1+ P(3,2) - P(2,2))/(P(3,2) * P(2,1));

PT = M31 + M13;
PD = abs(M31 - M13)/min(M13, M31);
Z1=['The passage times are ',num2str(M13),',',num2str(M31),' and totals ',num2str(PT),'.'];
disp(Z1)
\[ Z2 = ['\text{The PD is}', \text{num2str(PD)}, '.']; \]
\[ \text{disp}(Z2) \]

**Appendix D**

```matlab
function x = generate(initial, P, n)
    x(1) = initial;
    m = length(P);
    for i = 1:m
        for j = 1:m
            cum(i, j) = sum(P(i, 1:j));
        end
    end
    for i = 1:n-1
        u = rand;
        vec = sign(cum(x(i), :) - u);
        s = sum(vec);
        if prod(vec) == 0;
            x(i+1) = 0.5*(m-s) + 0.5;
        else
            x(i+1) = 0.5*(m-s) + 1;
        end
    end
end
```

**Appendix E**

For Chapter 5

Using the model, \( PT = \frac{1+d-b}{dg} + \frac{1+f-b}{fe} \)

The rate of change of PT as a changes is, \( \frac{\partial(PT)}{\partial a} = \frac{\partial}{\partial a}(PT) \)

\[ = \frac{\partial}{\partial a} \left( \frac{1+(1-a)-b}{(1-a)g} + \frac{1+f-b}{fe} \right) \text{ since } d = 1-a \]

\[ = \frac{\partial}{\partial a} \left( \frac{2-a-b}{g-ag} \right) + 0 \]

\[ = \frac{(g-ag)(-1) - (2-a-b)(-g)}{(g-ag)^2} \text{ by quotient rule} \]

\[ = \frac{-g+ag+2g-ag-bg}{(g-ag)^2} = \frac{(g-bg)}{(g-ag)^2} \]
\[ \frac{\partial (PT)}{\partial c} = \frac{\partial}{\partial c} (PT) \]

\[ = 0 + \frac{\partial}{\partial c} \left( \frac{2 - c - b}{e - ce} \right) \]

\[ = \frac{(e - ce)(-1) - (2 - c - b)(-e)}{(e - ce)^2} \]

\[ = \frac{ce - e + 2e - ce - be}{(e - ce)^2} = \frac{(e - be)}{(e - ce)^2} \]

\[ = \frac{e(1 - b)}{(e - ce)^2} = \frac{1 - b}{e(1 - c)^2} \]

\[ = \frac{1 - b}{ef^2} \]

The rate of change of PT as e changes is 

\[ \frac{\partial (PT)}{\partial e} = \frac{\partial}{\partial e} (PT) \]

\[ = \frac{\partial}{\partial e} \left( \frac{1 + d - b}{dg} + \frac{1 + f - b}{fe} \right) \]

\[ = \frac{\partial}{\partial e} \left( \frac{1 + d - (1 - e - g)}{dg} + \frac{1 + f - (1 - e - g)}{fe} \right) \]

\[ = \frac{\partial}{\partial e} \left( \frac{1 + d + e + g}{dg} + \frac{1 + f + e + g}{fe} \right) \]

\[ = \frac{\partial}{\partial e} \left( \frac{d + e + g}{dg} + \frac{f + e + g}{fe} \right) \]

\[ = \frac{1}{dg} + \frac{fe(1) - (f + e + g)f}{fe} \]

\[ = \frac{1}{dg} - \frac{f^2 + fg}{(fe)^2} \]

\[ = \frac{1}{dg} - \frac{f + g}{fe^2} \]

The rate of change of PT as g changes is 

\[ \frac{\partial (PT)}{\partial g} = \frac{\partial}{\partial g} (PT) \]

\[ = \frac{g(1 - b)}{(g - ag)^2} = \frac{1 - b}{g(1 - a)^2} \]

\[ = \frac{1 - b}{gd^2} \]
The rate of change of PT as b changes is

\[ \frac{\partial (PT)}{\partial b} = \frac{-fe}{dg} \]

Appendix F

Eigenvalue nature in relation to D and T
Appendix G

% A modification of Merrill (2010) generate code to observe sample path approach to limit distribution
function L2 = generate(initial,P,n)
v1=1;v2=0;v3=0 ; % Sample always start at 1
x(1) = initial;
L2(1)=0;
m=length(P);
for i=1:m
    for j=1:m
        cum(i,j)=sum(P(i,1:j));
    end
end
for i=1:n-1
    u=rand;
    vec=sign(cum(x(i),:)-u);
    s=sum(vec);
    if prod(vec)==0;
        x(i+1)=0.5*(m-s)+.5;
    else
        x(i+1)= 0.5*(m-s)+1;
    end
    if x(i+1)==2;
        v2=v2+1
    else
        v2=v2;
    end
    L2(i+1)=v2/(i+1)
end
end
plot(ans)