**Marquette University**

**e-Publications@Marquette**

***Mathematics Faculty Research and Publications/College of Arts and Sciences***

***This paper is NOT THE PUBLISHED VERSION;* but the author’s final, peer-reviewed manuscript.** The published version may be accessed by following the link in the citation below.

*Discrete Mathematics*, Vol. 342, No. 4 (April 2019): 1048-1055. [DOI](10.1016/j.disc.2018.12.015). This article is ©Elsevier and permission has been granted for this version to appear in [e-Publications@Marquette](http://epublications.marquette.edu/). Elsevier does not grant permission for this article to be further copied/distributed or hosted elsewhere without the express permission from Elsevier.

Maximizing and minimizing the number of generalized colorings of trees

John Engbers

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201, United States

Christopher Stocker

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201, United States

# Abstract

We classify the trees on vertices with the maximum and the minimum number of certain generalized colorings, including conflict-free, odd, non-monochromatic, star, and star rainbow vertex colorings. We also extend a result of Cutler and Radcliffe on the maximum and minimum number of existence [homomorphisms](https://www.sciencedirect.com/topics/mathematics/homomorphism) from a tree to a completely looped graph on vertices.

# Keywords

Vertex coloring, Extremal enumeration, Tree, Conflict-free coloring

# 1. Introduction

Let be a simple graph, and let be a coloring of the vertices of . A proper coloring of is a coloring so that no edge of is monochromatic. When colors are used (i.e. ) we will often refer to a proper coloring as a proper -coloring.

Other types of vertex colorings have recently been investigated. One such variation places the allowable colors as the vertices in a graph and joins two vertices of with an edge if those colors can appear across an edge in . For a given , an -coloring of , or graph [*homomorphism*](https://www.sciencedirect.com/topics/mathematics/homomorphism) from to , is a coloring of using the scheme from the graph ; more precisely, an -coloring of is a function so that if with , then . Notice that when , the complete graph on vertices, an -coloring of corresponds to a proper -coloring of . When is an edge with one looped endvertex, an -coloring of corresponds to an independent set, or stable set, in .

Finding an -coloring of a graph can be difficult, and so much recent research has investigated a related [extremal](https://www.sciencedirect.com/topics/mathematics/extremal) problem: Given a family of graphs , which has the largest (smallest, respectively) number of -colorings? An answer to this question produces bounds on the number of -colorings for any graph in , and also implies bounds on the probability that a random coloring of the vertices of from the vertices of will be an -coloring of . Some families that have been considered include [regular graphs](https://www.sciencedirect.com/topics/mathematics/regular-graph), graphs with fixed minimum degree, and graphs with a fixed number of edges. For results and conjectures on the extremal -coloring question in these families, we refer the reader to two surveys, [[5]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b5) and [[15]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b15), and the numerous references therein.

One specific family that will be applicable in this paper is the family of all -vertex trees, which will be denoted by . Extremal independent set counts in trees were first studied in [[11]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b11), while extremal -coloring counts in trees for all other have also been considered [[4]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b4), [[7]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b7), [[12]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b12). In particular, for any the star is always the tree with the largest number of -colorings, but interestingly the path is not always the tree with the smallest number of -colorings. See [[4]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b4) for more details, and [[7]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b7) for a class of such that the path is the tree with the smallest number of -colorings. Note that every tree on vertices has the same number of proper -colorings.

In this paper, we will focus on this extremal question for a number of types of colorings in the family of -vertex trees . Let and in denote the path and star on vertices, respectively.

# 2. Definitions and statements of results

In this section we define various types of colorings and state the corresponding [extremal](https://www.sciencedirect.com/topics/mathematics/extremal) results for those colorings, and in Section [3](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "sec3) we provide the proofs of these extremal results. We start with the notion of a conflict-free coloring.

## **Definition**

A *conflict-free coloring* of a graph with colors is a function such that for every there is a color occurring exactly once in . The number of conflict-free colorings of with colors is denoted .

This definition of conflict-free colorings of a graph is a special case of conflict-free colorings of a [hypergraph](https://www.sciencedirect.com/topics/mathematics/hypergraph), which was introduced in [[8]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b8), [[13]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b13). Considering conflict-free colorings with restrictions on paths instead of closed neighborhoods was studied in [[3]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b3). Finding the minimum number of colors needed to admit a conflict-free coloring, even for disks in the plane, is NP-complete [[8]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b8).

One application of conflict-free colorings occurs in frequency assignments for cellular networks. Here, the vertices represent base stations and the colors represent frequencies assigned to the base stations. For a client to receive a signal from a base station, they must tune to the signal from some (nearby) base station, and they require that signal to come from only one station in order to avoid signal interference. By representing spatially close base stations via edge adjacency, this frequency assignment problem is modeled by a conflict-free coloring of the associated graph. For a survey of conflict-free colorings and its applications, see [[14]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b14).

We determine the trees with the extremal number of conflict-free colorings.

## Theorem 2.1

*Let and . Then*

*Equality occurs in the upper bound if and only if .*

*When , equality occurs in the lower bound if and only if . When, equality occurs in the lower bound if and only if does not contain a non-trivial subtree with the property that each vertex in has exactly one neighbor outside of .*

We next define two colorings that are related to conflict-free colorings.

## Definition

An *odd coloring* of a graph with colors is a function such that for each there is a color occurring an odd number of times in . The number of odd colorings of with colors is denoted .

## **Definition**

A star rainbow coloring of a graph with colors is a function such that for each no color occurs more than once in . The number of star rainbow colorings of with colors is denoted

Odd colorings were first introduced in [[2]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b2). It is clear that every star rainbow coloring of is a conflict-free coloring of , and every conflict-free coloring of is an odd coloring of . We determine the trees with the extremal number of odd and star rainbow colorings.

## Theorem 2.2

*Let and . Then*

*Equality occurs in the upper bound if and only if .*

*When , equality occurs in the lower bound if and only if . When , equality occurs in the lower bound if and only if contains at most one vertex of even degree.*

## Theorem 2.3

*Let and . Then*

*Equality occurs in the lower bound if and only if and in the upper bound if and only if .*

Next we define a non-monochromatic coloring, and give the corresponding extremal result for trees.

## Definition

A *non-monochromatic coloring* of a graph with colors is a function such that for each there are at least two colors occurring in . The number of non-monochromatic colorings of with colors is denoted .

## **Theorem 2.4**

Let and . Then

Equality occurs in the lower bound if and only if and in the upper bound if and only if .

As these results show, the extremal trees are often or . It is tempting to conjecture that any color restriction on will cause to maximize the number of colorings among all trees; in other words, that a maximizing graph is independent of the color restriction on . The corresponding statement in the family of [regular graphs](https://www.sciencedirect.com/topics/mathematics/regular-graph) is in fact true: the regular graph that maximizes the number of colorings given by a restriction on the colors on or on is independent of the coloring lists [[6, Theorem 9]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b6). The situation for trees, however, is different. We next present a related coloring scheme where the maximizing graph for the number of these colorings is not or .

## **Definition**

A 2-strong-conflict-free coloring of a graph with colors is a function such that for each there are at least two colors that occur exactly once in . The number of 2-strong-conflict-free colorings of with colors is denoted .

The -strong-conflict-free colorings (in fact, -strong-conflict-free colorings) were originally studied in [[1]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b1) under the name -conflict-free colorings; see also [[14]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b14). For -strong-conflict-free colorings we have the following lower bound.

## Theorem 2.5

*Let and . Then*

*Equality occurs if and only if .*

Furthermore, for 2-strong-conflict-free colorings neither the path nor the star is the maximizer. Take and . Then it is easy to see that and (all colorings of are obtained by coloring the center of with one color, exactly one leaf with a second color, and the remaining leaves with the third color). But now consider the tree which is a balanced double star, i.e., it has two adjacent vertices and (the centers of the double star), each with two leaves. There are ways to have distinct colors on and , and ways to color each pair of leaves, noting that a leaf must have a different color from its neighbor. There are ways to have the same color on and , and ways to color each pair of leaves in this situation. Therefore . For general , it is not clear what the maximizing tree is in this case.

These results show that the extremal graphs in are not independent of the list restrictions, unlike the results for regular graphs. Note that in regular graphs all neighborhoods (closed neighborhoods, respectively) have the same size, whereas the size of a neighborhood (closed neighborhood, respectively) in a tree can vary significantly from vertex to vertex.

We also consider colorings where the color classes induce a forest of stars. Here the color restrictions are on paths in the tree.

## Definition

A *star coloring* of a graph with colors is a function such that (1) implies , and (2) for each (not necessarily induced) in maps onto at least three colors. The number of star colorings of with colors is denoted .

Star colorings were first introduced by Grünbaum in [[10]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b10), and [[9]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "b9) contains results on the star [chromatic number](https://www.sciencedirect.com/topics/mathematics/chromatic-number) of various families of graphs. We determine the trees with the extremal number of star colorings.

## **Theorem 2.6**

Let and . Then

Equality occurs in the lower bound if and only if and in the upper bound if and only if .

We remark that the path minimizes and the star maximizes the number of star colorings in , which differs from the corresponding extremal results for conflict-free, odd, star rainbow, and non-monochromatic colorings.

Our techniques also allow us to extend a result from [[6]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b6) for colors to arbitrary . In that paper, the concept of an existence [homomorphism](https://www.sciencedirect.com/topics/mathematics/homomorphism) is introduced and investigated.

## **Definition**

[[6]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b6)

Suppose that and are graphs with possibly having loops. We say that a map is an existence homomorphism if, for every , there exists a such that . We let be the number of existence homomorphisms from to .

In [[6]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b6) the authors consider , the completely looped graph on [isolated vertices](https://www.sciencedirect.com/topics/mathematics/isolated-vertex), and in this case an existence homomorphism from a graph to is a coloring of the vertices of with colors so that each color class has no isolated vertices (note that a color class may be empty). They show the following.

## **Theorem 2.7**

Cutler–Radcliffe [[6]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b6)

If is a tree on vertices, then

with equality if and only if

We generalize [Theorem 2.7](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.7) to colors and also find the minimizing tree.

## **Theorem 2.8**

Let and . Then

Equality occurs in the lower bound if and only if and in the upper bound if and only if .

It would be interesting to investigate the maximum and minimum number of these colorings for various other families of graphs, including regular graphs (see [[6, Proposition 18]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#b6) for maximizing over all -regular graphs), graphs with a fixed minimum degree, and graphs with a fixed number of edges.

# 3. Proofs

In this section, we present the proofs of the theorems stated in Section [2](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "sec2).

## 3.1. Odd colorings — Proof of [Theorem 2.2](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.2)

First, we prove the [extremal](https://www.sciencedirect.com/topics/mathematics/extremal) result for odd colorings. Recall that an odd coloring is a vertex coloring where for every vertex there is a color occurring an odd number of times in .

### Proof of Theorem 2.2

Let . Notice that if is a vertex such that is even, then by parity considerations some color will appear an odd number of times on . Also note that if is an uncolored leaf whose neighbor is colored, then there are at most ways to extend the odd coloring to (by considering only the restrictions on the leaf ). So by coloring all non-leaves first, we see that if has leaves then . Since is the unique tree with at most two leaves and in every non-leaf has even degree, the above observations show that is the unique tree maximizing the number of odd colorings, and .

Since every tree has proper colorings and every proper coloring is an odd coloring, we have . As has leaves, it also has at most odd colorings, and so .

The remainder of the proof characterizes the trees that minimize the number of odd colorings. To do so, we characterize the trees admitting an odd coloring which is not proper. Suppose there is a path in . When , we color the middle two vertices with color and the outer two vertices with color . Then we color each neighbor of these four vertices with color . From here, we iteratively complete the coloring by coloring an uncolored vertex that has a neighbor that is colored by assigning a color for that is distinct from the color on . Note that the set of colored vertices is always connected and so each uncolored vertex has at most one colored neighbor. By construction, the vertices on the path all have color appearing once on their closed neighborhood, and any vertex not on the path has the color on appearing once on . Since this produces an odd coloring which is not proper, when the only tree that minimizes the number of odd colorings is the star.

Finally, assume that . We show that there is an odd coloring of which is not proper if and only if has at least two vertices of even degree. Suppose first that has two vertices of even degree, and let and denote a pair of [distinct vertices](https://www.sciencedirect.com/topics/mathematics/distinct-vertex) with even degree that has the minimum positive distance between them. This implies that each vertex on the path between and has odd degree. We color the vertices of this path with color , and then iteratively properly color the rest of . If is not on the path, then the color on is distinct from all neighbors of and so appears once on . Since and have even degree, by parity considerations they have a color appearing an odd number of times on their closed neighborhoods. Finally, any other vertices on the path between and have exactly three vertices in their closed neighborhood with color . This exhibits an odd coloring of which is not proper.

Now suppose that there is at most one even degree vertex in , and suppose that there is a non-proper odd coloring of . Consider a maximal length monochromatic path in a non-proper coloring of . One of the endpoints of this monochromatic path, say , must have odd, and by [maximality](https://www.sciencedirect.com/topics/mathematics/maximality) has exactly two vertices receiving one color in . But as the number of vertices in is even it follows that has an even number of vertices colored with the other color, which contradicts the definition of odd coloring at vertex . Therefore there is no odd coloring which is not proper in a tree with at most one even degree vertex. □

## 3.2. Conflict-free and non-monochromatic colorings — Proofs of

[Theorems 2.1 and 2.4](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.1)

We now prove the extremal results for conflict-free and non-monochromatic colorings. Recall first that a conflict-free coloring is a vertex coloring where for every vertex

there is a color occurring exactly once in .

### **Proof of Theorem 2.1**

Let . For minimizing, notice that every proper coloring of is a conflict-free coloring of , and every conflict-free coloring of is an odd coloring of . All odd colorings of are proper, and so all conflict-free colorings of the star are proper. Furthermore, the proof of [Theorem 2.2](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#thm2.2) shows that any tree with a has a non-proper conflict-free coloring when .

For , we show that a tree has only proper conflict-free colorings if and only if does not contain a non-trivial subtree with the property that each vertex in has exactly one neighbor outside . Suppose that a tree has a non-proper conflict-free coloring, and consider a color class containing a non-trivial component. Notice that every vertex in that non-trivial component must have exactly one neighbor outside the component (which necessarily is colored with the other color). Conversely, if a tree has a non-trivial subtree in which each vertex in has exactly one neighbor outside , then we can color with color and iteratively properly color the rest of the tree to produce a non-proper conflict-free coloring of .

We now turn to the maximization question. First, notice that , , and . For with we denote the vertices as , and we enumerate the conflict-free colorings of by conditioning on whether or not and have the same color. If they do not, then deleting leaves a conflict-free coloring on the remaining path. If they do, then deleting and leaves a conflict-free coloring of the remaining path, since in this case must have a different color from .

Since in either case the only restriction for the color on is that it must differ from the color on , we have the [recurrence](https://www.sciencedirect.com/topics/mathematics/recurrence)

Similarly, we can condition on whether or not and have the same color, and from there properly color (giving choices for a color on these vertices). If , notice that there is a non-proper conflict-free extension in which and have the same color (from the case where and have distinct colors). This implies that

(1)

With these calculations in hand, we move to proving that . Note by the characterization of uniqueness for minimizing the number of conflict-free colorings we have for

and . We induct on to show that the path is the unique tree maximizing the number of conflict-free colorings. The base cases are trivial, so suppose that , , and is a tree on vertices.

Find a maximum length path in and let be a penultimate vertex on this path. Denote the leaves adjacent to by and let be the non-pendant neighbor of (here we use that ). Then [partition](https://www.sciencedirect.com/topics/mathematics/partition) the conflict-free colorings of based on whether and have the same color or not.

If they do not, we delete which results in a conflict-free coloring of the remaining tree. Since each must have a different color from the color on , letting there are at most conflict-free colorings of where the colors on and are not the same.

If the colors on and are the same, we delete , and , which results in a conflict-free coloring of the resulting tree . In this case we know that and must have the same color, and each must have a color differing from . This produces an upper bound of for the number of conflict-free colorings of where the colors on and are the same.

Putting these together and using the [inductive hypothesis](https://www.sciencedirect.com/topics/mathematics/inductive-hypothesis) along with [(1)](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "fd1), we have

Furthermore, the last inequality is an equality only when by [(1)](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#fd1). When , we have strict inequality in moving from the first line to the second line unless deleting a leaf and deleting a leaf plus its neighbor from leaves and , respectively, which implies that . □

The proof for non-monochromatic colorings is similar to conflict-free colorings. Recall that a non-monochromatic coloring is a vertex coloring where for every vertex there are at least two colors occurring in . We remark that every proper coloring is a non-monochromatic coloring.

### **Proof of Theorem 2.4**

We describe the changes needed to the proof of [Theorem 2.1](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#thm2.1). Notice that again a leaf must receive a different color from its neighbor, and so the only non-monochromatic colorings of the star are proper colorings. For uniqueness, suppose that there exists a in a tree . Coloring the two leaves of the path with color and the remaining two vertices with color , and then iteratively properly coloring the rest of the tree, produces a non-proper coloring of .

For maximizing, in we again condition on whether or not and have the same color. If they do not, then we delete ; if they do, then we delete and This gives

We also have , , and .

As before, for we also can condition on and having the same color or not, giving

We prove the result for trees by induction on , with the base cases trivial. With the above bounds in place, the induction proceeds exactly as in the proof of [Theorem 2.1](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#thm2.1). □

## 3.3. Star rainbow colorings — Proof of [Theorem 2.3](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.3)

Here we prove the extremal results for star rainbow colorings. Recall that a star rainbow coloring is a vertex coloring where for every vertex all colors occur at most one time in . Notice that all star rainbow colorings are proper colorings and that every coloring that uses a distinct color for each vertex is a star rainbow coloring.

### Proof of Theorem 2.3

By considering the center of , we see that each vertex in a star rainbow coloring of must have a different color. Therefore we have , which is the minimum number for any -vertex tree. Also, any tree containing a path admits a star rainbow coloring where the two ends of the have color and the remaining vertices have distinct colors. This shows that uniquely minimizes the number of star rainbow colorings.

Notice that we can obtain the count of the number of star rainbow colorings of a tree by iteratively coloring starting with a leaf. To color a vertex (adjacent to a colored vertex ), the color on must avoid all of the distinct colors appearing on the neighbors of that have already been colored.

Using we see that . If then has [a vertex of degree](https://www.sciencedirect.com/topics/mathematics/degree-of-a-vertex) at least three, and so by iteratively coloring from a leaf there is one vertex with at most color possibilities. This gives Therefore the unique tree the most number of star rainbow colorings is the path. □

Notice that the iterative coloring procedure in the proof of [Theorem 2.3](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#thm2.3) gives the number of star rainbow colorings for any tree. In particular, let be a tree with vertices, and let be a leaf of . Let be iteratively chosen vertices so that the induced graph on for each forms a tree. Then the number of star rainbow colorings of is given by

In words, when coloring vertex , we must avoid the color on its unique neighbor that already has a color, and also any colors appearing on a vertex that is in the closed neighborhood of .

## 3.4. 2-Strong-Conflict-Free Colorings — Proof of [Theorem 2.5](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.5)

Here we prove the lower bound for 2-strong-conflict-free colorings of a tree. We first recall that a 2-strong-conflict-free coloring is a vertex coloring where for every vertex

there are at least two colors that occur exactly once in .

### Proof of Theorem 2.5

Note that , since the 2-strong-conflict-free colorings of are exactly the star rainbow colorings of .

Now let be an -vertex tree, and root at a leaf. Color the root and its unique neighbor, and then iteratively color out so that each vertex receives a different color than the color on the two closest vertices on the path to the root. This produces a 2-strong-conflict-free coloring of and so . If is a vertex with degree at least three, then one of the two neighbors of that is not closest to the root can receive the same color as the neighbor of closest to the root. Therefore if then . □

## 3.5. Star colorings — Proof of [Theorem 2.6](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.6)

Here we prove the extremal result for star colorings. Recall that a star coloring is a vertex coloring that is proper and has no 2-colored path .

### Proof of Theorem 2.6

Star colorings are proper colorings with color restrictions on all subgraphs. Since is the unique tree that has no subgraph, all proper colorings of are star colorings, and every other tree admits a proper coloring that is not a star coloring. In particular, uniquely maximizes the number of star colorings.

We now turn to minimizing the number of star colorings. As in the proof of [Theorem 2.1](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#thm2.1), we first find a recursion for the number of star colorings of paths, and to start we have , , and . For suppose the path has vertices , and we partition the star colorings of based on whether and have the same color or not. If they do, then and must have different colors, and those colors must also differ from the color on . Deleting and produces a star coloring of , and furthermore any star coloring of this extends to star colorings of of this type.

If and have different colors, then we delete and are left with a star coloring of . But any star coloring of extends to star colorings of of this type, as and have different colors so we can color with any color that differs from those on and .

Putting this together, we have

We now show that . We induct on to show that the path is the unique tree minimizing the number of star colorings. The base cases are trivial, so suppose that , , and is a tree on vertices. By the uniqueness of the upper bound we have , so we may assume . Let be a vertex with at most one non-pendant neighbor (take, for example, to be a penultimate vertex in a maximal path). We let distinct vertices , , and be such that is a leaf and , and partition the star colorings of based on the colors on and .

If the colors on and are different, we delete . This gives a star coloring of . But all star colorings of of this type come from a star coloring of by giving a different color from and , so there are exactly colorings of where the colors on and are distinct.

If the colors on and are the same, we delete and the pendant neighbors of (including ). We recover all star colorings of of this type from a star coloring of the remaining tree by giving a color different from the colors on and , the color on , and all other pendants a color that differs from . Therefore there are exactly colorings of where the colors on and are the same.

The preceding arguments show

Now any star coloring of (with vertices ) is obtained by starting with a star coloring of (vertices ), and iteratively coloring the remaining vertices as a star coloring. Since a star coloring is a proper coloring, this implies that there are at most choices for a color on each of , which gives .

Therefore, we see that

This proves the inequality, and so we move now to the characterization of equality. We first argue that to have equality requires , which will follow from showing that for . Given a star coloring of , we use an iterative proper coloring of for the non-strict inequality. But if , one such coloring will have the same colors on and and will choose a color on that appears on . This creates a 2-colored in , and so is a proper coloring extension that is not a star coloring of . Therefore for .

Finally, if , then by induction we have equality only when and are and . respectively, which implies that . □

## 3.6. Existence Homomorphisms — Proof of [Theorem 2.8](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "thm2.8)

Here we prove the results about existence [homomorphisms](https://www.sciencedirect.com/topics/mathematics/homomorphism) to . Recall that an existence homomorphism from to is a vertex coloring of so that no color class contains an [isolated vertex](https://www.sciencedirect.com/topics/mathematics/isolated-vertex).

### Proof of Theorem 2.8

For minimizing, notice that each tree can be monochromatically colored. For , these are the only possible colorings, since a leaf must have the same color as its neighbor. And if a tree has a path , then coloring one leaf and its neighbor with color and the other leaf and its neighbor with color , and monochromatically coloring the vertices in each component of , we see that every has an existence homomorphism to (with ) that is not a monochromatic coloring of .

For maximizing, we again begin by giving a [recursive definition](https://www.sciencedirect.com/topics/mathematics/recursive-definition) for . As a leaf must have the same color as its neighbor, we have that and . Let have vertices , and consider an existence homomorphism from to . If the colors on and are the same, then this color must also be the color on , and we delete and obtain an existence homomorphism from the remaining path to . If the colors on and differ, then we delete and to obtain an existence homomorphism from the remaining path to . Since in this latter case the color on and has possibilities, this gives the recurrence

Given with we can condition on whether or not and have the same color. By monochromatically coloring , we see that

(2)

with strict inequality coming from a coloring of in which and have distinct colors (in the case where and have the same color).

Now we induct on to show that is the unique -vertex tree maximizing . Notice that by the minimizing result, we may assume that as for . Consider a maximum length path in and let be a penultimate vertex on this path. Denote the leaves adjacent to by (so that ) and let be a non-pendant neighbor of . Then we enumerate the existence homomorphisms to based on whether or not and have the same color. Note that the color on must be the same as the color on .

If and have the same color, delete and obtain a tree that has an existence homomorphism to . Each existence homomorphism of comes from one existence homomorphism of of this type and vice versa, and so there are existence homomorphisms of so that and have the same color.

By the remarks above, induction, and [(2)](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub" \l "fd2) we have

For equality, by [(2)](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#fd2) we have . Given this, by induction we have and , which implies that . □

# References

[[1]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb1) M. Abellanas, P. Bose, J. Garcia, F. Hurtado, M. Nicolas, P.A. Ramos, On properties of higher order delaunay graphs with applications, in: Proc. 21st European Workshop on Computational Geometry, 2005, pp. 9–11.

[[2]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb2) Cheilaris P., Keszegh B., Pálvölgyi D.**Unique-maximum and conflict-free coloring for hypergraphs and tree graphs** SIAM J. Discrete Math., 27 (4) (2013), pp. 1775-1787

[[3]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb3) Cheilaris P., Tóth G.**Graph unique-maximum and conflict-free colorings** J. Discrete Algorithms, 9 (3) (2011), pp. 241-251

[[4]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb4) Csikvári P., Lin Z.**Graph homomorphisms between trees** Electron. J. Combin., 21 (4) (2014) #P49

[[5]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb5) Cutler J.**Coloring graphs with graphs: a survey** Graph Theory Notes NY, vol. 63 (2012), pp. 7-16

[[6]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb6) Cutler J., Radcliffe A.J.**Counting dominating sets and related structures in graphs** Discrete Math., 339 (2016), pp. 1593--1599

[[7]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb7) Engbers J., Galvin D.**Extremal-colorings of trees and 2-connect ed graphs** J. Combin. Theory Ser. B, 122 (2017), pp. 800-814

[[8]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb8) Even G., Lotker Z., Ron D., Smorodinsky S.**Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks** SIAM J. Comput., 33 (2003), pp. 94-136

[[9]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb9) Fertin G., Raspaud A., Reed B.**Star coloring of graphs** J. Graph Theory, 47 (3) (2004), pp. 163-182

[[10]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb10) Grünbaum B.**Acyclic colorings of planar graphs** Israel J. Math., 14 (3) (1973), pp. 390-408

[[11]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb11) Prodinger H., Tichy R.**Fibonacci numbers of graphs** Fibonacci Quart., 20 (1982), pp. 16-21

[[12]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb12) Sidorenko A.**A partially ordered set of functionals corresponding to graphs** Discrete Math., 131 (1994), pp. 263-277

[[13]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb13) Smorodinsky S.**Combinatorial problems in computational geometry** (Ph.D. thesis) School of Computer Science, Tel-Aviv University (2003)

[[14]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb14) Smorodinsky S.**Conflict-free coloring and its applications** Bárány I., et al. (Eds.), Geometry – Intuitive, Discrete, and Convex, Bolyai Soc. Math. Stud. Springer, Vol 24 (2013), pp. 331-389

[[15]](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bb15) Zhao Y.**Extremal regular graphs: independent sets and graph homomorphisms** Amer. Math. Monthly, 124 (2017), pp. 827-843

# Note

[1](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#bfn1)Research supported by the Simons Foundation, United States grant [524418](https://www.sciencedirect.com/science/article/pii/S0012365X18304369?via%3Dihub#GS1).