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Independent Sets in -Vertex -Chromatic -Connected Graphs

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# Abstract

We study the problem of maximizing the number of independent sets in vertex chromatic  -connected graphs. First we consider maximizing the total number of independent sets in such graphs with n sufficiently large, and for this problem we use a stability argument to find the unique extremal graph. We show that our result holds within the larger family of vertex chromatic graphs with minimum degree at least , again for n sufficiently large. We also maximize the number of independent sets of each fixed size in vertex 3-chromatic 2-connected graphs. We finally address maximizing the number of independent sets of size 2 (equivalently, minimizing the number of edges) over all vertex chromatic  -connected graphs.

# Keywords

Coloring, Connectivity, Independent set, Extremal graph

# 1. Introduction and statement of results

Given a finite, simple graph , an independent set  is a subset of  so that if , then . The *size* of an independent set is . We will let  denote the number of independent sets in  and  denote the number of independent sets of size  in . The quantity  has also been called the *Fibonacci number* of  [17] and the *Merrifield–Simmons index* of  [16].

There has been a large number of papers devoted to finding the maximum and minimum values of  and  as  ranges over some family of graphs. For a sampling of these results, we refer the reader to two surveys [3], [18] and the references found therein.

A *proper vertex coloring* of a graph  is an assignment of a color to each vertex so that no edge is monochromatic. A graph  is *chromatic* if there exists a proper coloring using  colors but not one with  colors. We call  *critical* if it is chromatic and every proper subgraph of  is at most chromatic. Finally, a graph  is  -*connected* if  and any graph obtained by deleting fewer than  vertices is connected. Recently Fox, He, and Manners [7] proved an old conjecture of Tomescu by finding the vertex chromatic connected graph with the maximum number of proper vertex colorings using  colors.

The focus of this note is on maximizing  and  within the family of vertex chromatic  -connected graphs. When  and , the maximum number of independent sets, and independent sets of each fixed size , in these families was determined in [5]. Our first result generalizes this to maximizing  when  for  large. Before we state it, we first define the extremal graphs for the various values of  and . Recall that for graphs  and , the graph  has vertex set  and edge set . We denote the complete and empty graphs on  vertices by  and , respectively.

**Definition 1.1** Fix  and . For , let , and for  let . See Fig. 1.

So, for example, when  we have , and the fact that  for all vertex  -connected bipartite graphs , with equality if and only if , appears as Corollary 2.2 in [1] (recall that an  -connected graph has minimum degree at least ). When  and k>1 the graph  is formed from a clique with  pendants attached to one vertex in the clique, and the fact that  for all connected chromatic graphs , with equality if and only if , appears as Corollary 3 in [5]. (Viewing  naturally as  in the case where , the analogous result appears as Corollary 2 in [5].)



Fig. 1. The graph  for the two possibilities for  and .

We show that this result is in fact true for all , , and  large.

**Theorem 1.2** *Let*  *and*  *be fixed. If*  *and*  *is an* *vertex* *chromatic*  *-connected graph, then*

*with equality if and only if* *.*

In fact, we prove the following more general result, from which Theorem 1.2 follows as  -connected graphs have minimum degree at least .

**Theorem 1.3** *Let*  *and*  *be fixed. If*  *and*  *is an* *vertex* *chromatic graph with minimum degree at least* *, then*

*with equality if and only if* *.*

The proof of Theorem 1.3 uses a stability technique, and proceeds by showing that any graph  satisfying  must have a similar structure to  in that  must have a large complete bipartite subgraph  for some constant . It then breaks into two cases depending on the values of  and , where the count of  is driven by those independent sets that completely avoid the size  part of .

In [5], the authors ask what can be said in the family of vertex chromatic  -connected graphs for independent sets of size  when . We also provide some results here for specific values of , , and . The first new case is that of 2-connected 3-chromatic graphs; we remark that results on maximizing  over all vertex 2-connected graphs appear in [12].

**Definition 1.4** A *theta graph* joins vertices  and  with three internally disjoint paths of (edge) lengths , , and . We denote this graph by .

We have, for example, that , and it is apparent that . Note that when  is even, the corresponding  path has an odd number of internal vertices. We now state the theorem.

**Theorem 1.5** *Let* *, and let*  *be an* *vertex* 3*-chromatic* 2*-connected graph. Then we have the following:*

* *if*  *is odd, then*  *with equality if and only if* *;*
* *if*  *is even, then* *, where at least one of* *,* *, or*  *is even, with equality if and only if*  *where at least one of* *,* *, or*  *is even; and*
* *for all* *,* *, and for*  *we have equality if and only if* *.*

The results for maximizing  amongst 3-chromatic 2-connected graphs are consequences of [12], and we briefly discuss this at the beginning of Section 3. Also, as a fairly routine consequence of results for independent sets of size  in vertex  -connected graphs, we show the following.

**Theorem 1.6** *Let* *,* *, and*  *be fixed. If*  *is an* *vertex* *chromatic* *-connected graph and* *, then*

Finally, we also consider the problem of maximizing the number of independent sets of size  in chromatic  -connected graphs. Note that an independent set of size 2 induces an edge in the complement of the graph, so this problem is equivalent to minimizing the number of edges.

The problem of minimizing edges has been studied for several related families of graphs. The minimum number of edges in a chromatic graph is clearly . The minimum number of edges in  -connected graphs is  due to Harary [11]. The minimum number of edges in critical graphs was first studied by Dirac [4] and Gallai [8], [9] and subsequently in [13], [14], [15]. Minimizing edges in chromatic  -edge-connected graphs was considered in [2], where it was briefly noted that some of the bounds also hold for  -connected graphs.

We present two sharp bounds for the minimum number of edges in chromatic  -connected graphs for the case that . The first has the extra condition that  and our bound coincides exactly with the result in [2]. However, their proof relies on edge-connectivity. Furthermore, our techniques in the range  allow us to tackle the range  as well; this appears as an unsolved case in [2]. All remaining cases for minimizing edges in chromatic  -connected graphs follow similarly to [2], so we omit the results here.

**Theorem 1.7** *If*  *is a* *chromatic*  *-connected graph with*  *and*  *then*

*and this bound is sharp.*

**Theorem 1.8** *If*  *is a* *chromatic*  *-connected graph with*  *and*  *then*

*and this bound is sharp.*

In the rest of the paper we present the proofs of our results. We prove Theorem 1.3 in Section 2, and we consider 2-connected, 3-chromatic graphs in Section 3 where we also prove Theorem 1.6. Then we consider maximizing independent sets of size  in Section 4. Finally, in Section 5, we highlight some open questions related to the results in this paper.

# 2. Proof of Theorem 1.3

In this section we will prove Theorem 1.3, and to do so we will use the following results from [5].

**Theorem 2.1** [5] *Let*  *be an* *vertex* *chromatic graph. Then*

*with equality if and only if* *.*

**Theorem 2.2** [5] *Let*  *be an* *vertex* *chromatic graph with* d *components. Then*

*with equality if and only if* *.*

We now move on to the proof.

*Proof of Theorem 1.3*Suppose that  is an vertex chromatic graph with minimum degree at least  which satisfies . We investigate the structure of the graph G. First note that  holds for each  and .

**Step 1:** *Show that*  *cannot contain a large matching.*

Consider a maximum matching  in , and let  denote the size of this maximum matching. In any independent set, at most one endpoint of each edge in  is in the independent set, giving three possibilities across each edge in . Therefore, by only considering the restrictions on the edges in , we have

If , then  and so , which contradicts the assumption that . Therefore, we know that the maximum size of a matching  in  is at most .

**Step 2:** *Show that there is a constant*  *so that*  *contains*  *as a subgraph.*

Let  be a maximum matching. By Step 1, there are  vertices that are endpoints in ; call this set of vertices . The set  has size  and, by maximality of , must form an independent set. Since  has minimum degree at least , each vertex in  must have at least  neighbors in . The pigeonhole principle then produces some set  of size , with , having at least  common neighbors in  for some constant . Using that  when  and , we see that we can take . This shows that  contains a (not necessarily induced) subgraph .

**Step 3:** *Estimate the number of independent sets in*  *that include a vertex from* *.*

There are at most  ways to include at least one vertex from . Then none of the at least  common neighbors of  can be in the independent set, so this gives an upper bound of

(1)

independent sets that contain some vertex from .

**Step 4:** *Find an upper bound on the number of independent sets in* *.*

We have a bound from those independent sets that contain a vertex from  above in (1). Those that do not contain a vertex from  correspond to the independent sets in , the graph obtained by deleting . Note that . The chromatic number of  must be at most , and so if the chromatic number is  then by Theorem 2.1 we have at most  independent sets of this type, with equality if and only if  is a complete graph on m vertices with  isolated vertices. To compare these maximal values for various , note that for  we have

(2)

We now look at the two cases depending on the values of  and .

**Case 1** (): Suppose that . We first argue that the chromatic number of  must be . Since  is obtained from a graph with chromatic number  by deleting  vertices, it follows that the chromatic number of  is at least . For sake of contradiction, suppose  has chromatic number at least . Then by the bound from (2) the graph  has at most  independent sets of this type. Combining this with (1) gives

For  (recalling also that ), we have , which is a contradiction.

Thus we now know that the chromatic number of  is , and we next aim to show that  has many components. By Theorem 2.2 an -vertex chromatic graph with exactly d components has at most  independent sets; note that this is an increasing function of  Therefore, if  has at most  components, we have

For , we have , which is a contradiction.

Therefore, we know:

* the chromatic number of  is ;
* has  vertices; and
* has at least  components.

This is only possible if  has exactly  components, one component is , and the rest are isolated vertices. Therefore  must be the graph .

Now, recall that  has minimum degree at least  and chromatic number . The minimum degree condition forces each vertex in  to be adjacent to each isolated vertex in .

If some vertex in  is not adjacent to some vertex in the complete component of , then those two vertices can be assigned the same color in a proper coloring and this can easily be extended to a ()-coloring of the vertices of , which contradicts the assumption that  is chromatic. Therefore the vertices in  must form a dominating set in . Furthermore, they must all be adjacent, or a similar argument shows that  can be properly colored with at most  colors. Therefore  must be the graph .

**Case 2** (): Suppose now that . Recall that  is the graph obtained from  by deleting , and that in this case . First suppose that  contains some edge . At most one of the endpoints of  can be in an independent set, and so this combined with (1) gives

where the strict inequality holds for . This is a contradiction to the assumption on .

Therefore  must be the empty graph. As  has minimum degree , this means that each vertex in  is adjacent to each vertex in . Since  is chromatic, the induced graph on  must be ()-chromatic. Since all edges are present between  and , we have

Now,  has  vertices and chromatic number , and so we know from Theorem 2.1 that it has at most  independent sets, with equality if and only if . Therefore

with equality if and only if , which implies that .

# 3. 2-connected 3-chromatic and the Proof of Theorem 1.6

In this section we first completely classify the 2-connected 3-chromatic graphs that maximize the total count of independent sets and the total count of independent sets of each (non-trivial) fixed size.

## 3.1. Total count of independent sets

We first show that the result for the total number of independent sets is essentially a corollary to a result from [12]. There it is proved that if  is a 2-connected graph with , then  with equality if and only if  is  or . Since  is 3-chromatic and  is not, it follows that if  we have that  is the 2-connected 3-chromatic graph with the most number of independent sets. For , note that  is 3-chromatic and

so for  and  the characterization of equality implies that  is the (not necessarily unique) graph with the maximum number of independent sets.

## 3.2. Size t independent sets

Now we move to independent sets of size . For , we have the following results for 2-connected graphs and graphs with fixed minimum degree .

**Theorem 3.1** [6] *Let* *. For every* *, every* *vertex graph* G *with minimum degree at least* 2 *satisfies*

*For*  *and*  *we have equality if and only if*  *is* *, where*  *is any graph on two vertices.*

**Theorem 3.2** [10] *Let* *. For every* *, every* *vertex graph*  *with minimum degree at least*  *satisfies*

*and when* *,*  *is the unique extremal graph.*

*Proof of Theorem 1.5.* First, we analyze the case when ; here to maximize  we want to minimize . A 2-connected graph  has minimum degree at least 2, and so

with equality if and only if  is 2-regular, which implies that

The equality characterization for  odd follows readily as  is the only 2-connected 2-regular graph.

When  is even and  is 2-connected, we still have the bound of  with equality if and only if  is 2-regular if and only if . Since in this case  is not 3-chromatic, this shows that

which proves the inequality as for an vertex theta graph  we have . We now work on the cases for equality.

Using that  is not 2-regular, by the handshaking lemma there must be at least two vertices of degree at least 3, or one vertex of degree at least 4. A single vertex of degree 4 with the remaining vertices having degree 2 is not possible in a 2-connected , as the deletion of the degree 4 vertex disconnects the graph. So now suppose  has degree 3 vertices  and , and the remaining vertices of  have degree 2. If two of the edges out of  are on a cycle that misses , then again the deletion of  disconnects the graph, which contradicts that  is 2-connected. So each edge out of  must be on some path that ends at , which implies that  is a theta graph.

Given that  is a theta graph , we need the conditions on , , and  so that  is 3-chromatic. By coloring  and  with different colors and then the paths between them, we see that the chromatic number is 2 when , , and  are all odd. So at least one of , , and  must be even, and by parity considerations not all of , , and  are even, so one parameter is even and another is odd. These two paths form an odd cycle in the graph, which shows that  is indeed 3-chromatic. This implies the characterization of equality.

Now we consider . When  there are no vertex 3-chromatic graphs that have an independent set of size . Since  has minimum degree at least 2, Theorem 3.1 implies that for every  and , we have

with equality if and only if  is  or . Recalling that  is bipartite, this proves the result and the characterization of equality for .

## 3.3. Proof of Theorem 1.6

Suppose that . By Theorem 3.2,  is an  -connected graph that has the maximum number of independent sets of size . When , these independent sets consist of  vertices from the size  partition class. So if , then , and therefore  for all chromatic  -connected graphs, where , , and .

When , Theorem 3.2 gives that  is the *unique*  -connected graph with the maximum number of independent sets of size . Since for  we have  and for chromatic  -connected G we have , this shows that .

# 4. The case of : Minimizing edges

In this section, we consider maximizing the number of independent sets of size  in chromatic  -connected graphs. As previously mentioned, this problem is equivalent to minimizing the number of edges in such graphs, so our results and proofs are stated as such.

## 4.1. Proof of Theorem 1.7

We start by constructing a graph that achieves the bound. We make use of the vertex,  -connected Harary graph, which we denote by . To construct , place n vertices  in order around a circle and join each vertex to the  vertices closest to it in either direction. In the event that  is odd, then also join each vertex to the vertex directly opposite (or as opposite as possible when n is odd). The Harary graph  has  edges, which is the minimum number of edges over all graphs with the same number of vertices and connectivity [11].

Consider the disjoint union of the complete graph  and a Harary graph . Since we are assuming  and , we can choose  vertices  from  and  adjacent vertices  along the circle in  and connect these vertices via a matching, . Now starting with a terminal edge, w1w2, of the ℓ-vertex path in , we remove every other edge of the path. In the case where is odd and is also odd, then we use the one higher degree vertex from in our vertex path and delete both edges to the sides. Call this graph and let denote the subgraph induced by the vertices of . See Figure 2. The graph has

Edges and we claim this graph is chromatic and connected.

Now  is chromatic since the subgraph  requires  colors. We show the graph is also  -connected by showing that any two vertices  and  are connected by  disjoint paths. This is true if  and  both belong to the subgraph  since .

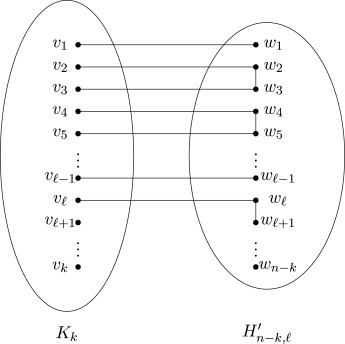


Fig. 2. The construction for  when  and  with  even.

Suppose  and  both belong to . By construction, the Harary graph  is  -connected, so there are  disjoint paths between any two vertices. We extend these disjoint paths to  over deleted edges, , by instead using edges , , and  in (with the obvious modification if and are odd and the degree vertex is internal on one of the paths). This the paths). This covers all cases except when itself has degree , in which case for some . But in this case since has degree , we can find disjoint paths between and any other vertex that excludes one of the edges or ; these disjoint paths can be extended as above.

Lastly, suppose  belongs to the  subgraph and  belongs to the  subgraph. From  there are  disjoint paths, whose internal vertices are in , with each ending at one of . Since  is -connected, from  there are  disjoint paths, whose internal vertices are in , with each ending at one of , and the same *ℓ* paths exist in . Therefore in all cases there are  disjoint paths in  between the vertices  and . Since deleting the endpoints of the matching in  disconnects the graph, this shows that  is  -connected.

*Proof of Theorem 1.7.*Sharpness follows from the graph . To prove the lower bound, let  be a chromatic  -connected graph. We will consider two cases: if  is critical and otherwise.

If G is critical, then all vertices have degree at least , so  has at least  edges. Then the difference in the number of edges between  and  is at least

Since , we have , and combining this with  gives . Thus,

and so the bound is correct in this case.

Suppose  is not critical. Then  has an (induced) critical subgraph. This subgraph, , has at least  vertices and minimum degree at least . Say  has  vertices. Then  has at least  edges. Consider the vertices in . Since  is  -connected, there must be at least  disjoint paths between  and . This requires at least  edges; assume there are  edges with one endpoint in  and the other endpoint in . Moreover, every vertex in  has to have minimum degree at least , since  is  -connected. This requires a minimum of an additional

edges with both endpoints in .

In total,  must have at least

edges. This bound is linear in x with positive slope , and so is minimized by the minimum value of . Since , we get

proving the claimed bound in this case. This finishes the proof.

## 4.2. Proof of Theorem 1.8

We again start by constructing a graph that achieves the bound. Consider the disjoint union of the complete graph  and the complete graph . Fix  vertices , and label the vertices in  by . For a fixed , add the  edges joining  and  for each  satisfying . Call this graph , and note that  has  edges. See Fig. 3.

We first claim that  is chromatic. It requires  colors on . And each vertex  can be colored with the color on  (where we consider  to be the vertex ). This is a coloring of .

Next, we claim that  is  -connected; note that removing  disconnects the graph (as ). We claim that any two vertices  and  are connected by  disjoint paths. This is clear if  and  are both in , since . It is also clear if  and  are both in , as there are  paths in , and the  edges to  from each vertex lead to  other edge disjoint paths. So assume  is some vertex in  and . We know that  has neighbors  for ; those have disjoint paths (with 0 or 1 edge) to . And furthermore the neighbors , , can use their edge to  and then edge  to produce the remaining disjoint paths from  to .

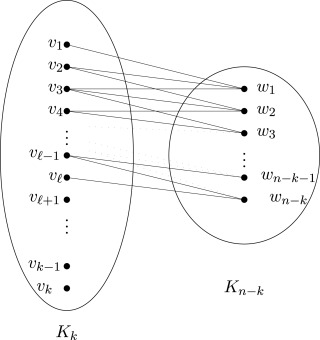


Fig. 3. The graph  when ; note that here  is adjacent to , , and .

*Proof of Theorem 1.8.*Note that we must have , since  implies , which contradicts the assumption of . Also note that the inequalities together imply that . Furthermore,  is not possible, as then  which is  connected, but  is not allowed. Therefore we can assume .

Sharpness comes from the graph . Suppose  is a chromatic  -connected graph with  and . We use a result from Gallai [9], which says that if  and , a critical graph on n vertices satisfies

there is a characterization of equality given in that paper as well. It is noted in various places that no critical graph has  vertices. We consider two cases:  is critical and otherwise.

If  is critical, then since  and our ranges of , , and  imply that , the result from Gallai [9] gives at least  edges. Then the difference between  and  is at least

Now  and  (as there is no critical graph on  vertices), and so the above expression is positive, which implies that bound is correct in this case.

Suppose now that  is not critical. Then  again has an induced critical subgraph  with  vertices. Since , if  then the Gallai bound applies, so  has at least  edges. If instead , the graph  has  edges. We cannot have  as there is no critical graph on  vertices.

Consider the vertices in . Each vertex must have at least  disjoint paths to , and so in particular must have minimum degree at least . Therefore the degree sum of the vertices in  is at least . There can be at most  edges that contribute two to this degree sum, coming from edges with both endpoints in . This means there must be at least  edges between  and . And if there are p edges missing inside the induced subgraph on , where , then we have at least  edges between  and .

Therefore the total number of edges in  is at least

where  is the indicator function on the event . Minimizing this, we take , giving at least

edges. The expression

is a quadratic function of (real-valued)  with leading coefficient . By the Extreme Value Theorem, the minimum occurs over  at  or . We also need to compare these values to , the other possible value for . (We remark that we separate out the case when  as the indicator function is not continuous, and so we cannot apply the Extreme Value Theorem on .) When  we have the bound

We need to show that when  or , we have a larger bound. For , the bound is

and when  the bound is

Computing the  count minus the  count gives

Now for this bound we have , so  means , or , so the difference in terms is positive in this case.

Computing the  count minus the  count gives

here  and  (as ). This shows the difference is positive in this case. Therefore, the minimum value occurs for , proving the claimed bound.

# 5. Concluding remarks

In this section we highlight a few open problems that are related to the contents of this paper. In Theorem 1.3, we characterized the vertex chromatic  -connected graph with the maximum number of independent sets for large . We expect the result to hold for all n for which the graph  is chromatic and  -connected.

**Conjecture 5.1** *Let*  *and* *. If*  *is an* *vertex* *chromatic*  *-connected graph, then*.

**Conjecture 5.2** *Let*  *and* *. If*  *is an* *vertex* *chromatic*  *-connected graph, then* .

Conjecture 5.2 is true for  and  (for ), as was shown in Section 3. There are also open questions related to the number of independent sets of size . We expect the following to hold, which extends Theorem 1.6 down to .

**Conjecture 5.3** *Let*  *and* *. If*  *is an* *vertex* *chromatic*  *-connected graph and* *, then*

It is also natural to conjecture that this behavior holds for  as well.

**Conjecture 5.4** *Let*  *and* *. If*  *is an* *vertex* *chromatic*  *-connected graph and* *, then*

These last two conjectures appeared as questions in [5]. We note that the corresponding results for  and  are shown in Theorem 1.5, and that the cases when  and  appear in [5].

# Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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