**Marquette University**

**e-Publications@Marquette**

***Mathematics and Statistical Sciences Faculty Research and Publications/College of Arts and Sciences***

***This paper is NOT THE PUBLISHED VERSION*.**

Access the published version via the link in the citation below.

*Pakistan Journal of Statistics and Operation Research*, Vol. 17, No. 1 (2021): 227-234. [DOI](https://doi.org/10.18187/pjsor.v17i1.3702). This article is © University of the Punjab and permission has been granted for this version to appear in [e-Publications@Marquette](http://epublications.marquette.edu/). University of the Punjab does not grant permission for this article to be further copied/distributed or hosted elsewhere without the express permission from University of the Punjab.

Remarks on and Characterizations of 2S-Lindley and 2D-Lindley Distributions Introduced by Chesneau et al. (2020)

G.G. Hamedani

Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI

Mahrokh Najaf

Department of Electrical Engineering, University of Science and Culture, Tehran, Iran

# Abstract

Chesneau et al. (2020) considered the distributions of sum and differences of two independent and identically distributed random variables with the common Lindley distribution. They derived, very nicely, the above mentioned distributions and provided certain important mathematical and statistical properties as well as simulations and applications of the new distributions. In this short note, we like to show that the assumption of “independence” can be replaced with a much weaker assumption of “sub-independence”. Then we present certain characterizations of the proposed distributions to complete, in someway, their work.

# Key Words:

Lindley Distribution, Independence, Sub-Independence, Identically Distributed Random Variables, Characterizations of Distributions

# 1. Introduction

To make this short note self-contained, we will copy some parts of our previous work Hamedani (2013) here. We may in some occasions have asked ourselves if there is a concept between “uncorrelatedness” and “independence” of two random variables. It seems that the concept of “sub-independence” is the one: it is much stronger than uncorrelatedness and much weaker than independence. The notion of sub-independence seems important in the sense that under usual assumptions, Khintchine’s Law of Large Numbers and Lindeberg-Levy’s Central Limit Theorem as well as other important theorems in probability and statistics hold for a sequence of sub-independence (*s.i.*) random variables. While sub-independence can be substituted for independence in many cases, it is difficult, in general, to find conditions under which the former implies the latter. Even in the case of two discrete identically distributed random variables X and Y, the joint distribution can assume many forms consistent with sub-independence.

Limit theorems as well as other well-known results in probability and statistics are often based on the distribution of the sums of independent (and often identically distributed) random variables rather than the joint distribution of the summands. Therefore, the full force of independence of the summands will not be required. In other words, it is the convolution of the marginal distributions which is needed, rather than the joint distribution of the summands which, in the case of independence, is the product of the marginal distributions. The concept of sub-independence is shown to be sufficient to yield the conclusions of these theorems and results. This is precisely the reason for the statement: “why assume independence when you can get by with sub-independence.”

The concept of sub-independence can help to provide solution for some modeling problems where the variable of interest is the sum of a few components. Examples include household income, the total profit of major firms in an industry, and a regression model where and *ε* are uncorrelated; however, they may not be independent. For example, in Bazargan et al.(2007), the return value of significant wave height (*Y*) is modeled by the sum of a cyclic function of random delay *,* and a residual term *.* They found that the two components are at least uncorrelated, but not independent and used sub-independence to compute the distribution of the return value.

Let *X* and *Y* be two random variables with joint and marginal cumulative distribution functions (*cdf s*) and respectively. Then *X* and *Y* are said to be independent if and only if

(1*.*1)

or equivalently, if and only if

(1*.*2)

where *,*  and *,* respectively, are the corresponding joint and marginal *cfs*. Note that (1*.*1) and (1*.*2) are also equivalent to

(1*.*3)

The concept of sub-independence, as far as we have gathered, was formally introduced by Durairajan (1979) and developed by Hamedani in the past 40 years, stated as follows: The random variables *X* and *Y* with *cdf s* and are sub-independent (*s.i.*) if the *cdf* of is given by

(1*.*4)

or equivalently if and only if

(1*.*5)

The drawback of the concept of sub-independence in comparison with that of independence has been that the former does not have an equivalent definition in the sense of (1*.*3) *,* which some believe, to be the natural definition of independence. We found such a definition which is stated below. We shall give the definition for the continuous case (Definition 1.1).

We observe that the half-plane can be expressed as a countable disjoint union of rectangles:

where and are intervals. Now, let be a continuous random vector and for , let

and

**Definition 1.1**. The continuous random variables *X* and *Y* are *s.i.* if for every

To see that (1*.*6) is equivalent to (1*.*4), observe that (*LHS* of (1*.*6))

(1*.*7)

where *.* Now, if *X* and *Y* are *s.i.* then

where *,* are probability measures on R defined by

and is the product measure. We also observe that (*RHS* of (1*.*6))

(1*.*8)

Now, (1*.*7) and (1*.*8) will be equal if , which is true since the points in are obtained by shifting each point in *H* over to the right by units and then up by units.

If *X* and *Y* are *s.i.*, then unlike independence, *X* and *αY* are not necessarily *s.i.* for any real . This demonstrates how weak is the concept of sub-independence in comparison with that of independence. Please observe the following simple example.

/

**Example 1.1**. Let *X* and *Y* have the joint *cf* given by

where is an appropriate constant. (The characteristic function is the Fourier Transform of probability density function (*pdf*), so the corresponding joint *pdf* is given by

where .

Then *X* and *Y* are *s.i.* standard normal random variables, and hence is normal with mean 0 and variance 2, but *X* and −*Y* are not *s.i.* and consequently *X* − *Y* does not have a normal distribution.

The concept of sub-independence defined above can be extended to random variables as follows.

**Definition 1.2**. The random variables are s.i. if for each subset of

# 2. Remarks

i) If the random variables *X* and *Y* are sub-independent identically distributed (*s.i.i.d.*) with the common Lindley distribution with the parameter , the characteristic function of is

The *cf* of *X* is

and since *X* and *Y* are *s.i.,* we have

ii) If the random variables *X* and *Y* are identically distributed (*i.d.*) with the common Lindley distribution with the parameter , and if *X* and −*Y* are *s.i.*, the characteristic function of is

The *cf* of , under the assumption of *s.i.* of *X* and −*Y* , is

iii) For a detailed treatment of the concept of sub-independence, we refer the interested reader to Hamedani (2013).

# 3. Characterizations of the 2S-Lindley and 2D-Lindley Distributions

Chesneau et al. (2020) introduced the distributions of the sum and differences of two *i.i.d.* (now, *s.i.i.d.*) Lindley random variables with the parameter *θ >* 0 (called 2S-Lindley and 2D-Lindley) with their respective *pdf s* given by

(3*.*1)

and

(3*.*2)

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words, we need to have certain conditions under which we may be able to recover the probability law of the data. So, the characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated moments. This type of characterization initiated by Galambos and Kotz (1978) and followed by other authors such as Kotz and Shanbhag( 1980), Glänzel et al. (1984), Glänzel(1987), Glänzel and Hamedani (2001) and Kim and Jeon (2013), to name a few. For example, Kim and Jeon (2013) proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present characterizations of the 2S-Lindley and 2D-Lindley distributions based on the conditional expectation (truncated moments) of certain functions of the random variable.

We will employ Theorem 1 of Glänzel (1987) given in the Appendix A. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

**Proposition 3.1**. Let *X* be a continuous random variable and let and for *.* Then *X* has *pdf* (3*.*1) if and only if the function defined in Theorem 1 is of the form

**Proof**. If *X* has pdf (3*.*1), then

and

and hence

We also have

Conversely, if is of the above form, then

and

Now, according to Theorem 1, *X* has density (3*.*1) *.*

Corollary 3.1. Suppose *X* is a continuous random variable. Let be as in Proposition 3.1. Then *X* has density (3*.*1) if and only if there exist functions and defined in Theorem 1 for which the following first order differential equation holds

Corollary 3.2. The differential equation in Corollary 3.1 has the following general solution

where *D* is a constant.

**Proof**. If *X* has pdf (3*.*1), then clearly the differential equation holds. Now, if the differential equation holds, then

or

or

from which we arrive at

A set of functions satisfying the above differential equation is given in Proposition 3.1 with *.* Clearly, there are other triplets satisfying the conditions of Theorem 1.

**Remark 3.1**. Similar results can be stated for the 2D-Lindley distribution as well.

# References

Bazargan, H., Bahai, H., and Aminzadeh-Gohari, A. (2007). Calculating the return value using a mathematical model of significant wave height. *J. Marine Science and Technology*, 11:34–42.

Chesneau, C., Tomy, L., and Gillatiose, J. (2020). On a sum and difference of two lindley distributions: theory and applications. *Revstat-Statistical Journal*, 18(5):673–695.

Galambos, J. and Kotz, S. (1978). *Characterizations of probability distributions: A unified approach with an emphasis on exponential and related Models, Lecture Notes in Mathematics, Vol. 675*. Springer, Berlin.

Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. *Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986)*, B:75–84.

Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments, statistics:. *A Journal of Theoretical and Applied Statistics*, 21(4):613–618.

Glänzel, W. and Hamedani, G. G. (2001). Characterizations of the univariate continuous distributions. *Studia Scientiarum Mathematicarum Hungarica*, 37(1-2):83–118.

Glänzel, W., Teles, A., and Schubert, A. (1984). Characterization by truncated moments and its application to Pearson-type distributions. *Zeitschrift fu¨r Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66:173–182.

Hamedani, G. (2013). Sub-independence: an expository perspective. *CSTM,*, 42(20):3615–3638.

Kim, J. H. and Jeon, Y. (2013). Credibility theory based on trimming. *Insurance: Mathematics and Economics*, 53(1):36–47.

Kotz, S. and Shanbhag, D. N. (1980). Some new approaches to probability distributions. *Advances in Applied Probability*, 12(4):903–921.

# Appendix A

Theorem 1. Let be a given probability space and let be an interval for some might as well be allowed)*.* Let be a continuous random variable with the distribution function and let and be two real functions defined on *H* such that

is defined with some real function *ξ*. Assume that and is twice continuously differentiable and strictly monotone function on the set *H*. Finally, assume that the equation has no real solution in the interior of *H*. Then *F* is uniquely determined by the functions *,*  and , particularly

where the function *s* is a solution of the differential equation and is the normalization constant, such that .

Note: The goal is to have the function as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see  [Glänzel](#bookmark0) (1990)) , in particular, let us assume that there is a sequence of random variables with distribution functions such that the functions and satisfy the conditions of Theorem 1 and let for some continuously differentiable real functions and *.* Let, finally, *X* be a random variable with the distribution *F.* Under the condition that and are uniformly integrable and the family is relatively compact, the sequence converges to *X* in distribution if and only if converges to , where

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions *,*  and , respectively. It guarantees, for instance, the ’convergence’ of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if .

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions *,*  and, specially, should be as simple as possible. Since the function triplet is not uniquely determined, it is often possible to choose as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take *q*1 (*x*) ≡ 1*,* which reduces the condition of Theorem 1 to *.* We, however, believe that employing three functions *q*1*, q*2 and *ξ* will enhance the domain of applicability of Theorem 1.