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Characterizations and Inﬁnite Divisibility of Certain Recently Introduced Distributions IV

G.G. Hamedani

Department of Mathematics, Statistics and Computer Science, Marquette University, USA

# Abstract

Certain characterizations of recently proposed univariate continuous distributions are presented in diﬀerent directions. This work contains a good number of reintroduced distributions and may serve as a source of preventing the reinvention and/or duplication of the existing distributions in the future.

# Keywords

Univariate continuous distributions, Hazard function, Reverse hazard function, Characterizations

# 1. Introduction

This work is a continuation of our previous works (Hamedani and Safavimanesh, 2016) , (Hamedani, 2017) and (Hamedani, 2018) on characterizations and inﬁnite divisibility of distributions introduced in 2016-2018. The current work and our previous published papers mentioned above may serve as a source of preventing the reinvention and/or duplication of the existing distributions in the future. As pointed out in our papers, a good number of proposed distributions have already been introduced in the literature. We believe the authors should do a detailed literature search before spending time on the already existing distributions. In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model ﬁts the requirements of a speciﬁc underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Thus, the problem of characterizing a distribution is an important problem in various ﬁelds and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in diﬀerent directions. This work deals with various characterizations of Erlang-Lindley (EL) distribution of Abd El-Monsef et al.; Exponentiated Transmuted Weibull Geometric (ETWG) distribution of Fattah et al.; Transmut-ed Generalized Exponential (TGE) of Khan et al.; Truncated Inverted Generalized Exponential (TIGE) distribution of Genc¸; Transmuted Generalized Inverted Exponential (TGIE) distribution of Elbatal; Kumaraswamy GEV (KumGEV) distribution of Eljabari and Nadarajah; Generalized Exponential-G (GEG) distribution of Maurya et al.; Type II Odd Lindley Exponential (TIIOLE) distribution of Korkmaz and Yousof; Burr-X Exponentiated Fr´echet (BXEF) distribution of Zayed and Butt; Exponentiated Lomax Geometric (ELG) distribution of Hassan and Abd-Allah; Exponential Pareto Power Series (EPPS) distribution of Elbatal et al.; Transmuted Weibull Fr´echet (TWFr) distribution of Ahsan ul Haq et al.; Extended Burr XII (EBXII) distribution of Abouelmagd et al.; Generalized Weibull Burr XII (GWBXII) distribution of Maksaei and Altun; Burr Type XII (BTXII) distribution of Kumar; Gamma Burr XII (GBXII) of Guerra et al.; Generalized Marshall-Olkin-Kumaraswamy-G (GMOKw-G) family of distributions of Chakraborty and Handique; Weibull Burr X (WBX) distribution of Ibrahim et al.; Kumaraswamy Transmuted Pareto (KwTP) distribution of Chhetri et al.; Odd Log-Logistic Logarithmic Generated (OLLL-G) family of distributions of Alizadeh et al.; Marshall-Olkin Extended Inverse Pareto (MOEIP) distribution of Gharib et al.; Marshall-Olkin Extended Inverse Weibull (MOEIW) distribution of Okasha et al.; Upper Truncated Lindley (UTL) distribution of Singh et al.; Transmuted Exponentiated Fr´echet (TEF) distribution of Elbatal et al.; Generalized Inverted Generalized Exponential (GIGE) distribution of Oguntunde and Adejumo; Transmuted Rayleigh (TR) distribution of Ahmad et al.; McDonald Quasi Lindley (McQL) distribution of Roozegar and Esfandiyari; Geometric Weibull Poisson (GWP) distribution of Mansour and Abd Elrazik; SS Transformation of Expo-nential (SSTE) distribution of Kumar et al.; Generalization of the BurrXII-Poisson (GBXIIP) distribution of Muhammad; Log-Logistic Generated Weibull (LLGW) distribution of Abdel-Hamid and Albasuoni; Odd Generalized Exponentiated Linear Failure Rate (OGELFR) distribution of El-Damcese et al.; Transmuted Exponentiated Inverse Rayleigh (TEIR) distribution of Ahsan ul Haq; Transmuted Two-Parameter Lindley (TTL) distribution of Al-Khazaleh et al.; Transmuted Janardan (TJ) distribution of Al-Omari et al.; Exponentiated Generalized Weibull-Gompertz (EGWG) distribution of El-Bassiouny et al.; Complementary Exponentiated BurrXII Poisson (CEBXIIP) distribution of Muhammad; Kumaraswamy Power Function (Kw-PF) distribution of Abdul-Moniem; Odd Burr III-G (OBIII-G) family of distributions of Jamal et al.; Odd Generalized Exponential Generalized Linear Exponential (OGE-GLE) distribution of Luguterah and Nasiru; Exponentiated Inverse Flexible Weibull Extension (EIFWE) distribution of El-Morshedy et al.; Transmuted Weighted Exponential (TWE) distribution of Dar et al.; Beta Transmuted Pareto (BTP) distribution of Chherti et al.; Generalized Burr-G (GBG) family of distributions of Nasir et al.; Exponentiated Weibull-Power Function (EWPF) distribution of Hassan and Assar; Transmuted Two-Parameter Lindley (TTL1) distribution of Kemaloglu and Yilmaz; Marshall-Olkin Log-Logistic Extended Weibull (MOLLEW) distribution of Lepetu et al.; Modiﬁed Slash Birnbaum-Saunders (MSBS) distribution of Reyes et al.; Inverse Power Lindley (IPL) distribution of Barco et al.; Weibull Weibull (WW) distribution of Abouelmagd et al.; Beta Generated Kumarswamy-G (BGKw-G) family of distributions of Handique et al.; New Life Model (NLM) of Muhammad; Marshall-Olkin Exponentiated Burr XII (MOEBXII) distribution of Cordeiro et al.; Beta Nadarajah-Haghighi (BNH) distribution of Dias et al.; New Two-Parameter Weibull (NTPW) distribution of Rasekhi et al.; Generalized Half-t (GHT) distribution of Bulut et al.; Gamma Extended-G (GE-G) distribution of Cordeiro et al.; Modiﬁed Behrens-Fisher (MB-F) distribution of Nadarajah and Li; Maxwell-Weibull (M-W) distribution of Sharma et al.; Exponentiated Weibull-Exponential (EWE) distribution of Elgarhy et al.; α Logarithmic Transformed (αLT) family of distributions of Dey et al.; Topp-Leone Odd Log-Logistics (TLOLL) family of distributions of Brito et al.; Two-Parameter Maxwell (TP-M) distribution of Dey et al.; Generalized Quadratic Hazard Rate (GQHR) distribution of Sarhan; New Four-Parameter Weibull (NFPW) distribution of Yousof et al.; Odd Lindley Burr XII (OLBXII) distribution of Abouelmagd et al.; Dagum Poisson (DP) distribution of Oluyede et al.; Burr XII Modiﬁed Weibull (BXIIMW) of Mdlongwa et al.; Burr X Pareto (BXP) of Korkmaz et al.; General Class of Flexible Weibull (GCFW) distributions of Park and Park; Power Lomax Poisson (PLP) distribution of Hassan and Nassr; Modiﬁed Weibull Poisson (MWP) distribution of Ghorbani et al.; Exponentiated Power Lindley Geometric (EPLG) distribution of Alizadeh et al.; Generalized Modiﬁed Weibull Power Series (GMWPS) distribution of Bagheri et al.; Additive Modiﬁed Weibull Odd Log-Logistic-G (AMWOLLG) family of distributions of Ghorbani Et al.; Exponentiated Power Lindley Power Series (EPLPS) distribution of Alizadeh et al.; Weibull-R (W-R) family of distributions of Ghosh and Nadarajah; Wrapped Lindley (WL) distribution of Joshi and Jose; Beta Weibull-G family of distributions of Makubate et al.; Transmuted Weibull Regression (TWR) distribution of Granzotto et al.. These characterizations are presented in diﬀerent directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) in terms of the reverse (reversed) hazard function and (iv) based on the conditional expectation of certain function of the random variable. Note that (i) can be employed also when the cd f (cumulative distribution function) does not have a closed form. In deﬁning the above distributions we shall try to employ the same parameter notation as used by the original authors. We follow the same order as listed above.

1) The cd f of EL is given by

(1)

, where and are parameters and

2) The cd f of ETWG is given by

(2)

where are all positive, and p are parameters.

3) The *cd f* of TGE is given by

(3)

where both positive and are parameters.

**Remark 1.** The TGE distribution of Khan et al. (2017b) is a special case of TEAW distribution of Nofal et al. (2017),

which has already been characterized in Hamedani (2017a). We believe that Khan et al. were not aware of Nofal et al.’s paper.

4) The *cd f* of TIGE is given by

(4)

where are positive parameters and

Remark 2. The TIGE distribution of Genc¸ (2017) is a special case of INGIW distribution of Khan et al. (2017b), which has already been characterized in Hamedani (2017b). We believe that Genc¸ was not aware of Khan et al.’s paper.

5) The *cd f* of TGIE is given by

(5)

where both positive and are parameters.

Remark 3. The TGIE distribution of Elbatal (2013) is a special case of TNGIW distribution of Khan et al. (2017a), which has already been characterized in Hamedani (2017b).

6) The *cd f* of KumGEV (WLOG, for ) is given by

(6)

where are all positive parameters.

7) The cd f of GEG is given by

(7)

where is a parameter and are cd f and pd f (probability density function) of the baseline distribution. Remark 4. One may add another parameter to GEG and express (7) as

8) The cd f of TIIOLE is given by

(8)

where is a parameter.

**Remark 5.** The TIIOLE distribution of Korkmaz and Yousof (2017) is a special case of OL-G distribution of Gomez-Silva et al. (2017), which has been characterized in Hamedani (2017a).

9) The cd f of BXEF is given by

(9)

where are all positive parameters.

**Remark 6.** The BXEF distribution of Zayed and Butt (2017) is a submodel of BX-G distribution introduced by Yousof et al. (2016), which has been characterized in Hamedani (2017a).

10) The cd f of ELG is given by

(10)

where all positive and are parameters.

11) The cd f of EPPS is given b

(11)

where are all positive parameters and

**Remark 7**. The EPPS distribution of Elbatal et al. (2017) is a submodel of RGTLPS distribution introduced by Condino and Domma (2016), which has been characterized in Hamedani (2017a).

12) The cd f of TWFr, for , is given by

(12)

where all positive and are parameters.

13) The *cd f* of EBXII is given by

(13)

where are all positive parameters.

**Remark 8.** The EBXII distribution of Abouelmagd et al. (2017) is a minor extension of FPBXII distribution introduced by Afify et al. (2017), which has been characterized in Hamedani (2017a).

14) The *cd f* of GWBXII is given, by the authors as

.

It is easy to see that the cd f can be written as

.

and WLOG, we can set .

Therefore, the *cd f* of GWBXII (WLOG, we can set ) is given by

(14)

where are all positive parameters.

15) The cd f of BTXII is given by

(15)

where are all positive parameters.

**Remark 9.** The BTXII distribution of Kumar (2017) is a special submodel of TBTXII distribution introduced by Al-Khazaleh (2016), which has been characterized in Hamedani (2017a).

16) The *cd f* of GBXII is given by

(16)

where are all positive parameters and

**Remark 10.** The GBXII distribution of Guerra et al. (2017) is a special submodel of G-P distribution introduced by

Alzaatreh et al. (2012) as well as a special submodel of GEW distribution of Cordeiro et al.(2016). Both distributions (G-P and GEW) have been characterized in the Research Monograph by Hamedani and Maadooliat (2017).

17) The *cd f* of GMOKw-G is given by

(17)

where are all positive parameters and and is the baseline *cd f* with corresponding *pd f*

**Remark 11.** A slightly more general case of the GMOKw-G distribution has been characterized in Hamedani (2017a).

18) The *cd f* of WBX is given by

(18)

where are all positive parameters.

19) The *cd f* of KwTP is given by

(19)

where all positive and are parameters.

**Remark 12.** The distribution has been considered by several authors before. In fact a more general case was proposed by Mahmoud et al. (2015). For , the *cd f* (37) was taken up by Bourguignon et al. (2013).

20) The *cd f* of OLLL-G is given by

(20)

where are parameters and is the baseline *cd f* with corresponding *pd f* .

**Remark 13.** The Zografos-Balakrishnan Odd Log-Logistic (ZBOLL-G) distribution with *cd f*

where was proposed by Cordeiro et al. (2016). The OLLL-G distribution seems to be a simple variation of ZBOLL-G, which has been characterized in the Research Monograph by Hamedani and Maadooliat (2017).

21) The *cd f* of MOEIP is given by

(21)

, where are positive parameters.

**Remark 14.** in equation (6) of Gharib et al. (2017) should be replaced with .

22) The *cd f* of MOEIW is given by

(22)

where are positive parameters.

**Remark 15.** The MOEIW distribution was introduced under the name of Marshall-Olkin Fr´echet (M-OF) distribution by Krishna et al. (2013). The latter has been characterized in the Research Monograph by Hamedani and Maadooliat (2017).

23) The *cd f* of UTL is given by

(23)

where are positive parameters.

24) The *cd f* of TEF, is given by

(24)

, where positive and are parameters.

25) The *cd f* of GIGE is given by

(25)

where are positive parameters.

Remark 16. The GIGE distribution is a special case of, at least two distributions. We mention here, the Exponentiated-Exponential Fr´echet (EEFr) distribution of Mansoor et al. which has been characterized in the Research Monograph by Hamedani and Maadooliat (2017).

26) The *cd f* of TR is given by

(26)

where are positive parameters.

**Remark 17.** The TR distribution is a special case of the Kumaraswamy-Transmuted Exponentiated Modified Weibull (Kw-TEMW) distribution of Al-babtain et al. (Communications in Statist. Theory-Methods, forthcoming) which has been characterized in Hamedani (IJSP, forthcoming).

27) The *cd f* of McQL is given by

(27)

where all positive and are positive parameters.

**Remark 18**. For the McQL distribution of Roozegar and Esfandiary (2015) reduces to the Kumaraswamy Quasi Lindley (KQL) distribution of Elbatal and Elgarhy (2013). The KQL distribution has characterized in the Research Monograph by Hamedani and Maadooliat (2017).

28) The *cd f* of GWP is given by

(28)

where all positive and are parameters.

29) The *cd f* of SSTE is given by

(29)

where is a parameter.

30) The *cd f* of GBXIIP is given by

(30)

where are positive parameters.

31) The *cd f* of LLGW is given by

(31)

where are positive parameters.

**Remark 19.** The LLGW distribution of Abdel-Hamid and Albasuoni (2016) is a special submodel of TBTXII distribution introduced by Al-Khazaleh (2016), which has been characterized in Hamedani (2017a). Furthermore, LLGW distribution reduces to BTXII distribution for , please see Remark 9.

32) The *cd f* of OGELFR is given by

(32)

where are positive parameters.

**Remark 20.** The OGELFR distribution of El-Damcese et al. (2016) is a special submodel of Exponentiated Weibull

Rayleigh (EWR) distribution of Elgarhy, which has been characterized in Hamedani (2017a).

33) The *cd f* of TEIR is given by

(33)

where are positive and are parameters.

**Remark 21.** The TEIR distribution of Ahsan ul Haq (2016) is a special submodel of Transmuted Kumaraswamy Exponentiated Inverse Rayleigh (TKEIR) distribution of Badr (2017), which has been characterized in Hamedani (2017a).

34) The *cd f* of TTL is given by

(34)

where are parameters.

35) The *cd f* of TJ is given by

(35)

where are parameters.

**Remark 22.** The TJ distribution of Al-Omari et al. (2017) is similar to the Maxwell Length Biased (MLB) distribution of Saghir et al. (2016) whose *cd f* is . The MLB distribution has been characterized in Hamedani (2017a).

36) The *cd f* of EGWG is given by

(36)

where are all positive parameters.

**Remark 23.** The EGWG distribution of El-Bassiouny et al. (2017) is the same as the one proposed by El-Damcese et al. (2015). The EGWG distribution is in turn a special case of EWR distribution of Elgarhy (2015) which has been characterized in Hamedani (2017a).

37) The *cd f* of CEBXIIP is given by

(37)

where are all positive parameters.

**Remark 24.** The CEBXIIP distribution of Muhammad (2017) is a special case of Poisson-G (Po-G) family of distributions of Abouelmagd et al. (2017) if one takes in the formula for the Po-G distribution. The Po-G distribution has been characterized in Hamedani (2018).

38) The *cd f* of Kw-PF is given by

(38)

where are all positive parameters.

39) The *cd f* of OBIII-G is given by

(39)

where are all positive parameters and is the base *cd f* with corresponding *pd f* .

**Remark 25.** The OBIII-G distribution of Jamal et al. (2017) is the same as the one proposed by Arifa et al. (2017), called Modified Burr III G (MBIIIG) distribution, which has been characterized in Hamedani (2017a).

40) The *cd f* of OGE-GLE is given by

(40)

where are all positive parameters.

**Remark 26.** The OGE-GLE distribution of Luguterah and Nasiru (2017) is a special case of the distribution of Abdelall (2016), called Odd Generalized Exponential Modified Weibull (OGEMW) distribution, which has been characterized in Hamedani and Safavimanesh (2017).

41) The *cd f* of EIFWE is given by

(41)

where are all positive parameters.

**Remark 27**. For , the EIFWE distribution of El-Morshedy et al. (2017) is the same as the one proposed by the same authors (listed in the different order) in (2015), called Inverse Flexible Weibull Extension (IFWE) distribution. The IFWE has been characterized in a Research Monograph by Hamedani and Maadooliat (2017).

42) The *cd f* of TWE is given by

(42)

where and are parameters.

43) The cd f of BTP is given by

(43)

where and are parameters.

**Remark 28.** For , the BTP distribution of Chhetri et al. (2017) reduces to Generalized Beta Exponentiated Pareto (GBEP) distribution of Mead (2014). The GBEP distribution has been characterized in the Research Monograph by Hamedani and Maadootiat (2017).

44) The *cd f* of GBG is given by

(44)

where *c*, *k* are positive parameters and is the baseline *cd f* with the corresponding *pd f* , which may depend on the vector parameter .

45) The *cd f* of EWPF is given by

(45)

, where , *a* are all positive parameters.

**Remark 29.** For , the EWPF distribution of Hassan and Assar (2017) reduces to Weibull Power Function (WPF) distribution of Tahir et al. (2014). The WPF distribution has been characterized in the Research Monograph by Hamedani and Maadootiat (2017).

46) The *cd f* of TTL1 is given by

(46)

, where are parameters.

**Remark 30**. The TLL1 distribution of Kemaloglu and Yilmaz (2017) is the same as the TTL distribution of Al-Khazaleh et al. (2016). The TTL distribution is numbered 34) in the present work. TTL1 (TTL) has been characterized in Hamedani (2017a).

47) The *cd f* of MOLLEW is given by

(47)

where are parameters and is a non-negative differentiable function with, which may depend on the parameter .

48) The *cd f* of MSBS is given by

(48)

, where are positive parameters, is *pd f* of standard normal distribution and

49) The *cd f* of IPL is given by

(49)

where are positive parameters.

**Remark 31.** The IPL distribution of Barco et al. (2017) is the same as the Generalized Inverse Lindley (GIL) distribution of Asgharzadeh et al. (2016). The GIL distribution has been characterized in Hamedani (2017a).

50) The *cd f* of WW is given by

(50)

where are positive parameters.

**Remark 32.** The *cd f* is of the form (50), which has been characterized in Hamedani (2017a). Similarly, *cd f s* and which have been characterized in the Research Monograph by Hamedani and Maadooliat (2017).

51) The *cd f* of BGKw-G is given by

(51)

, where are positive parameters.

**Remark 33**. The *cd f*  where is an absolutely continuous *cd f* has been introduced before and has been characterized in Hamedani (2016). Letting in we arrive at (101).

52) The *cd f* of NLM is given by

(52)

where is a positive parameter.

53) The *cd f* of MOEBXII is given by

(53)

where are all positive parameters and .

**Remark 34.** The cd f where baseline cd f was introduced by Dias et al. (2016). Letting , renaming as (for ) and we arrive at (53). Thus, *cd f* (53) is a special case of Dias et al. distribution.

54) The *cd f* of BNH is given by

(54)

, where are all positive parameters.

**Remark 35.** As pointed out in Remark 33, the *cd f* where isan absolutely continuous *cd f* has been introduced before and has been characterized in Hamedani (2016). Letting in , we arrive at (54).

55) The *cd f* of NTPW is given by

(55)

where are all positive parameters.

**Remark 36.** The *cd f*  give in (55) has been characterized in Hamedani (2017a) under the name BrX-W. 56) The *pd f* of GHT is given by

(56)

where are all positive parameters, and

57) The cd f of GE-G is given by

(57)

where are all positive parameters and is the baseline *cd f* with corresponding *pd f* , depending on the parameter vector .

58) The *cd f* of MB-F is given by

for , and

and

for and respectively, where are all positive parameters, We will take up here the more interesting case of . Similar conclusions can be made for the above two cases as well. The *cd f* of MB-F, for the case of , is given by

(58)

where

59) The cd f of M-W is given by

(59)

where are positive parameters.s

60) The *cd f* of EWE is given by

(60)

where are positive parameters.

**Remark 37.** The *cd f*  give in (60) is a special case of the Exponentiated Weibull Rayleigh (EWR) distribution of Elgarhy. The EWR distribution has been characterized in Hamedani (2017a).

61) The *cd f* of LT are given by

(61)

where is a parameter and is the baseline *cd f* with *pd f*

62) The *cd f* of TLOLL is given by

where are positive parameters and is the baseline *cd f* with *pd f*  which depends on the parameter vector .

**Remark 38.** The *cd f*  given above is similar to the New Family (NF) of distributions of Alizadeh et al.

(2015). The *cd f* of NF distribution is given by

(62)

The NF distribution has been characterized in the Research Monograph by Hamedani and Maadooliat(2017). 63) The *cd f* of TW-M is given by

(63)

where and are parameters.

64) The *cd f* of GQHR is given by

(64)

where and are parameters.

**Remark 39**. The *cd f*  given in (64) is a special case of the Exponentiated Generalized (EG) class of

distributions of Cordeiro et al. (2013). The *cd f* of EG distribution is given by

where is the baseline distribution. Taking we arrive at (64) .The EG distribution has been characterized in Hamedani (2016). The *cd f* given in (64) is also a special case of the Kumaraswamy Quadratic Hazard Rate (KQHR) distribution of Elbatal and Aryal (2013). The KQHR has been characterized in Hamedani (2017).

65) The *cd f* of NFPW is given by

(65)

where and are parameters.

**Remark 40.** The *cd f* given in (65) is a special case of the Kumaraswamy-Tansmuted Exponentiated Modified Weibull (KwTEMW) distributions of Al-babtain et al. (2015). The *cd f* of KwTEMW distribution is given by

, where are parameters. Taking and , we arrive at (65) .The KwTEMW distribution has been characterized in Hamedani (2018).

66) The cd f of OLBXII is given by

(66)

, where are positive parameters.

67) The *cd f* of DP is given by

(67)

where are all positive parameters.

68) The *cd f* of BXIIMW is given by

(68)

where all positive and are parameters.

69) The *cd f* of BXP is given by

(69)

where are positive parameters.

**Remark 41.** The *cd f* given in (69) is a special case of the Exponentiated Weibull-Pareto (EW-P) distributions of Afify et al. (2016). The *cd f* of EW-P distribution is given by

where are positive parameters. Taking , we arrive at (69) .The EW-P distribution has been characterized in Hamedani and Maadooliat (2017).

70) The *cd f* of GCFW is given by

(70)

where and are parameters, are non-negative and nondecreasing for all with and

71) The *cd f* of PLP is given by

(71)

where and are positive parameters.

**Remark 42.** The *cd f* given in (71) is a special case of the Truncated Weibull-G (TW-G) distributions of Najarzadegan et al. (2017). The TW-G distribution of Najarzadegan et al. is not new either since this distribution was introduced by Gomes et al. (2015), which is characterized in Hamedani and Maadooliat (2017).

72) The *cd f* of MWP is given by

(72)

where are parameters.

73) The *cd f* of EPLG is given by

(73)

where are parameters.

74) The *cd f* of GMWPS is given by

(74)

where and are parameters and are nonnegative real numbers.

**Remark 43.** The *cd f* given in (146) is a special case of the general form of

mentioned in Tahmasebi and Jafari (2016) in which they set . Taking in the above , yields (74). The distribution of Tahmasebi and Jafari (2016) is characterized in Hamedani and Safavimanesh (2017b).

75) The *cd f* of AMWOLLG (WLOG, for ) is given by

(75)

where and are parameters.

76) The *cd f* of EPLPS is given by

(76)

where are parameters and are nonnegative real numbers.

77) The cd f of W-R is given by

(77)

where are parameters and are *cd f* and *pd f* of the special distribution taken in Ghosh and Nadarajah (2018).

78) The *cd f* of WL is given by

(78)

, where is a parameter.

79) The cd f of BW-G is given by

(79)

where are positive parameters and are *cd f* and *pd f* of the baseline distribution which depends on the parameter vector .

**Remark 44**. The *cd f* , where is an absolutely continuous *cd f* has been introduced before and has been characterized in Hamedani (2016). Letting in , we arrive at (79).

80) The *cd f* of TWR is given by

(80)

where are parameters and as a parameter depending on a covariate vector given by .

# 2. Characterizations of Distributions

We present our characterizations in four subsections.

## 2.1 Characterizations Based on Two Truncated Moments

This subsection deals with the characterizations of distributions listed in Section 1 based on the ratio of two truncated moments. Our first characterization employs a theorem due to Gl¨anzel (1987), see Theorem 1 of Appendix A . The result, however, holds also when the interval is not closed, since the condition of the Theorem is on the interior of

**Proposition 1.1.** Let be a continuous random variable and let and for . Then for , the random variable as *cd f* (1) if and only if the function defined in Theorem 1 is of the form

Proof. Suppose the random variable has *cd f* (1), then

and

Further,

Conversely, if is of the above form, then

and consequently

Now, according to Theorem 1, has *cd f* (1) .

**Corollary 1.1.** Let be a continuous random variable and let be as in Proposition 1.1. For , the random variable has *cd f* (1) if and only if there exist functions and defined in Theorem 1 satisfying the following differential equation

**Remark 1.1.** The general solution of the differential equation in Corollary 1.1 is

whereis a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1.1 with . Clearly, there are other triplets which satisfy conditions of Theorem1.

Clearly, a Proposition, a Corollary and a Remark similar to the Proposition 1.1, Corollary 1.1 and Remark 1.1 can be

stated for each of the distributions mentioned in the Introduction. For each of these distributions, we give below, the

functions and corresponding to Theorem 1.

**2.**  , and for

**6.** andfor

**7.**  and for

**10.** and for

**12.** and for

**14.**  and for

**18.**  and for

**19.**  and for

**21.**  and for

**23.**  and for

**24.**  and for

**28.**  and for

**29.**  and for

**30.**  and for

**38.**  and for

**42.**  *and for*

**43**. and for

**44.**  and for

**47.**  and for

**48.**  and for

**50.**  and for

**52.**  and for

**55**. and for

**56.**  and for

**57.** and for and any positive number.

**58**. and for

**60.**  and for

**62.**  and for

**65.**  and for

**66**. and for

**67.**  *and for*

**69**. and for

**71.**  and for

**72.**  and for

**74.**  and for

**75.**  and for

**76**. and for

**77.**  and for

**78**. and for

## 2.2 Characterization in Terms of Hazard Function

The hazard function, , of a twice differentiable distribution function, , satisfies the following first order differential equation It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of EL (for ), ETWG (for ), KumGEV, GEG, ELG (for ), TWFr (for ), GWBXII, WBX, KwTP, MOEIP, UTL, TEF (for ), GBXIIP (for ), TTL (for ), Kw-PF, GBG, WW, NLM, TL, OLBXII, BXIIMW, GCFW, MWP, AMWOLLG, EPLPS, W-G and TWR distributions in terms of the hazard function, which are not of the above trivial form.

**Proposition 2.1**. Let be a continuous random variable. The random variable has *cd f* (1) (for ) if and only if its hazard function satisfies the following differential equation.

Proof. If has *cd f* (1) for , then clearly the above differential equation holds. If the differential equation holds, then

from which we arrive at the hazard function corresponding to the *cd f* (1) .

A Proposition similar to that of Proposition 2.1 will be stated (without proof) for each one of the distributions listed in subsection 2.1.

**Proposition 2.2**. Let be a continuous random variable. The random variable has *cd f* (2) (for ), if and only if its hazard function satisfies the following differential equation

**Proposition 2.3.** Let be a continuous random variable. The random variable has *cd f* (6), if and only if its hazard function satisfies the following differential equation

**Proposition 2.4**. Let be a continuous random variable. The random variable has cd f (7), if and only if its hazard function satisfies the following differential equation

**Proposition 2.5.** Let be a continuous random variable. The random variable has *cd f* (10), for , if and only if its hazard function satisfies the following differential equation

**Proposition 2.6.** Let be a continuous random variable. The random variable has cd f (12), for , if and only if its hazard function satisfies the following differential equation

**Proposition 2.7.** Let be a continuous random variable. The random variable has *cd f* (14), if and only if its hazard function satisfies the following differential equation

**Proposition 2.8.** Let be a continuous random variable. The random variablehas *cd f* (18), if and only if its hazard function satisfies the following differential equation

**Proposition 2.9.** Let be a continuous random variable. The random variable has *pd f* (38), if and only if its hazard function satisfies the following differential equation

**Proposition 2.10**. Let be a continuous random variable. The random variable has *cd f* (21), if and only if its hazard function satisfies the following differential equation

**Proposition 2.11.** Let be a continuous random variable. The random variable has *pd f* (46), if and only if its hazard function satisfies the following differential equation

**Proposition 2.12.** Let be a continuous random variable. The random variable has *cd f* (24), for , if and only if its hazard function satisfies the following differential equation

**Proposition 2.13.** Let be a continuous random variable. The random variable has *cd f* (30), for , if and only if its hazard function satisfies the following differential equation

**Proposition 2.14.** Let be a continuous random variable. The random variable has cd f (34), for , if and only if its hazard function satisfies the following differential equation

**Proposition 2.15.** Let be a continuous random variable. The random variable has *cd f* (38), if and only if its hazard function satisfies the following differential equation

**Proposition 2.16.** Let be a continuous random variable. The random variable has cd f (44), if and only if its hazard function satisfies the following differential equation

**Proposition 2.17**. Let be a continuous random variable. The random variable has cd f (50), if and only if its hazard function satisfies the following differential equation

**Proposition 2.18**. Let be a continuous random variable. The random variable has *pd f* (104), if and only if its hazard function satisfies the following differential equation

**Proposition 2.19.** Let be a continuous random variable. The random variable has *cd f* (60), if and only if its hazard function satisfies the following differential equation

**Proposition 2.20**. Let be a continuous random variable. The random variable has cd f (65), if and only if its hazard function satisfies the following differential equation

**Proposition 2.21.** Let be a continuous random variable. The random variable has *cd f* (67) if and only if its hazard function satisfies the differential equation

**Proposition 2.22.** Let be a continuous random variable. The random variable has *cd f* (69) if and only if its hazard function satisfies the differential equation

**Proposition 2.23.** Let be a continuous random variable. The random variable has *cd f* (71) if and only if its hazard function satisfies the differential equation

**Proposition 2.24.** Let be a continuous random variable. The random variable has *cd f* (74) if and only if its hazard function satisfies the differential equation

**Proposition 2.25**. Let be a continuous random variable. The random variable has *cd f* (75) if and

only if its hazard function satisfies the differential equation

**Proposition 2.26.** Let be a continuous random variable. The random variable has *cd f* (76) if and only if its hazard function satisfies the differential equation

**Proposition 2.27.** Let be a continuous random variable. The random variable has *cd f* (79) if and only if its hazard function satisfies the differential equation

## 2.3 Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, , of a twice differentiable distribution function, , is defined as In this subsection we present characterizations of ETWG (for ), GEG, ELG (for ), KwTP (for ) MOEIP, SSTE, GBXIIP, OLBXII, MWP, EPLG distributions in terms of the reverse hazard function.

**Proposition 3.1**. Let be a continuous random variable. The random variable has pd f (4) (for ) if and only if its reverse hazard function satisfies the following differential equation

Proof. If has *pd f* (4) for , then clearly the above differential equation holds. If the differential equation holds, then

from which we arrive at the hazard function corresponding to the *cd f* (2) .

A Proposition similar to that of Proposition 3.1 will be stated (without proof) for each one of the distributions listed in subsection 3.1.

**Proposition 3.2.** Let be a continuous random variable. The random variable has *cd f* (7) if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.3.** Let be a continuous random variable. The random variable has *pd f* (20), for , if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.4.** Let be a continuous random variable. The random variable has *cd f* (19), for , if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.5**. Let be a continuous random variable. The random variable has *cd f* (21), if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.6.** Let be a continuous random variable. The random variable has *cd f* (29), if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.7.** Let be a continuous random variable. The random variable has *cd f* (30), if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.8.** Let be a continuous random variable. The random variable has cd f (65), if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.9.** Let be a continuous random variable. The random variable has *cd f* (71), if and only if its reverse hazard function satisfies the following differential equation

**Proposition 3.10**. Let be a continuous random variable. The random variable has *cd f* (72), if and only if its reverse hazard function satisfies the following differential equation

## 2.4 Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function (or ) of and characterize the distribution of in terms of the truncated moment of (or ). The following propositions have already appeared in Hamedani’s previous work (2013), so we will just state them here which can be used to characterize some of the distributions listed in Section 1.

**Proposition H1.** Let be a continuous random variable with cd f . Let be a differentiable

function on with Then for ,

if and only if

**Proposition H2**. Let be a continuous random variable with cd f . Let be a differentiable

function on with Then for ,

implies

**Remarks 4.1.** (*A*) For and , Proposition H1 provides a

characterization of KumGEV distribution. (*B*) For and , Proposition H1 provides a characterization of GWBXII distribution. (*C*) For and , Proposition H1 provides a characterization of WBX distribution. (*D*) For and , Proposition H1 provides a characterization of KwTP distribution. (*E*) For and , Proposition H1 provides a characterization of TEF distribution. (*F*) For and , Proposition H2 provides a characterization of GBXIIP distribution. (*G*) For and , Proposition H1 provides a characterization of TLL distribution. For and , Proposition H1 provides a characterization of Kw-PF distribution. (*I*) For and , Proposition H1 provides a characterization of GBG distribution. For and , Proposition H1 provides a characterization of WW distribution. For and , Proposition H1 provides a characterization of NLM distribution. For and , Proposition H1 provides a characterization of OLBXII distribution For and , Proposition H1 provides a characterization of BXIIMW distribution. For and , Proposition H1

provides a characterization of W-R distribution.

# 3. Infinite Divisibility

Bondesson (1979) showed that all the members of the following families

(B1)

(B2)

(B3)

(B4)

where the natural restrictions are put on the unspecified parameters, are infinitely divisible. The last one is the lognormal density.

**Remark 3.1.** Bondesson (1992, Theorem 6.2.4) pointed out that multiplying densities (B1) − (B4) by for and , will result in densities which are also infinitely divisible.

**Remark 3.2.** The distributions, listed in Section 1, whose densities can be expressed, in view of Remark 3.1, in the form (B1) are: BTXII , ZBXII (for ), MOEIP (for ), LLGW and MB-F (for ) ; in the form (B2) are: EBXII (for ) , McQL (for ); in the form of (B3) are: TIGE for , TGIE (for ), MOEIW (for ) and TEF (for ).

# 4. Concluding Remarks

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will vitally depend on the characterizations of the selected distribution. A good number of recently introduced distributions which have important applications in many different fields have been mentioned in this work. Certain characterizations of these distributions have been established. We hope that these results will be of interest to the investigators who may believe their models have distributions mentioned here and are looking for justifying the validity of their models. It is known that determining a distribution is infinitely divisible or not via the existing representations is not easy. We have used Bondesson’s classifications to show that some of the distributions taken up in this work are infinitely divisible. This could be helpful to some researchers. Finally, we like to mention that the distributions mentioned in this work may be a source of preventing duplications of the existing distributions.

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# Appendix A

Theorem 1. Let be a given probability space and let be an interval for some might as well be allowed) . Let be a continuous random variable with the distribution function and let and be two real functions defined on such that

is defined with some real function . Assume that and is twice continuously differentiable and strictly monotone function on the set . Finally, assume that the equation has no real solution in the interior of . Then is uniquely determined by the functions and , particularly

where the function is a solution of the differential equation and is the normalization constant, such that

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