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ABSTRACT. This paper will impose limits on the possible sets of irreducible character degrees of a normally monomial 5-group of maximal class.

1. Introduction

Let G be a finite p-group. Then G is an M-group ("monomial") which means that every irreducible ordinary character of G can be induced from a linear character of some subgroup. If one can always choose the subgroup from which one is inducing the linear character to be normal, then we say G is an nM-group ("normally monomial"). Recently some papers ([**3**], [**6**]) have studied the character degrees of normally monomial p-groups and especially, normally monomial p-groups of maximal class.

In this paper we will prove:

THEOREM. *Let G be a normally monomial, maximal class 5-group. Then* $\mathrm{cd}(G)$ *is either* $\{1, 5, 25, 5^4\}$*, the set of all powers of 5 up to some limit,* $\{1, 5, 25, \dots, 5^k\}$ *with* $k \geq 1$*, or either of those two forms with degree 25 removed.*

Throughout the paper, the computer algebra system Magma [**2**] was used to gain insight, verify computations, and compute required small cases.

REMARK. In computations of character degrees of all maximal class 5-groups of order up to 5^{13} and some up to 5^{15}, no groups have been found with character degrees of the form $\{1, 5, 25, 5^4\}$ or $\{1, 5, 5^4\}$. Furthermore, all these groups have character degrees $\{1, 5, 5^3\}$ or $\{1, 5, 25, \dots, 5^k\}$ even when having nilpotence class of P_1 greater than 2 or G not normally monomial.

2. Certain Module Homomorphisms

In this paper, we are only concerned with $p = 5$. Nonetheless, we will occasionally use "p" for "5" in order to make certain formulas more readable or familiar.

We need to set up the machinery from [**5**, Section 8.2]. Let K be the 5th local cyclotomic number field and $\mathcal{O}$ be the ring of integers in K.

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In $\mathcal{O}$, let θ be a fixed primitive 5th root of unity. Note that multiplication by θ is an additive automorphism of $\mathcal{O}$ which has order 5. Thus we can use this action to view $\mathcal{O}$ as a C_5-module.

Let $\kappa = \theta - 1$ and $\mathfrak{p} = (\kappa)$. Then $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$, $|\mathfrak{p}^i : \mathfrak{p}^{i+1}| = p$ for all $i \geq 1$, and $\mathfrak{p}^{p-1} = (p)$.

DEFINITION 2.1. For any $\zeta \in \mathcal{O}$, define the κ-weight of ζ to be the smallest positive integer i such that $\zeta \notin \mathfrak{p}^i$. For instance, units of $\mathcal{O}$ have κ-weight 1.

The patterns of commutators in our groups will be closely related to homomorphisms from $\mathcal{O} \wedge \mathcal{O}$ to $\mathcal{O}$, so we wish to examine some of these maps. For any integer a coprime to 5, define σ_a to be a ring automorphism of $\mathcal{O}$ which maps θ to θ^a. We then define $S_2 : \mathcal{O} \wedge \mathcal{O} \to \mathcal{O}$ by

$$(x \wedge y)S_2 = (x\sigma_2)(y\sigma_{-1}) - (y\sigma_2)(x\sigma_{-1}).$$

We would also like to define $\kappa_a = \kappa\sigma_a$ and u_a to be the unit in $\mathcal{O}$ such that $\kappa_a = \kappa u_a$.

LEMMA 2.2. *The value* $(\kappa^{j+1} \wedge \kappa^j)S_2$ *has* κ*-weight* $2j+2$, $j \geq 0$.

PROOF. We compute

$$\begin{aligned}(\kappa^{j+1} \wedge \kappa^j)S_2 &= \kappa_2^{j+1}\kappa_{-1}^j - \kappa_2^j\kappa_{-1}^{j+1} \\ &= \kappa^{2j+1}(u_2^{j+1}u_{-1}^j - u_2^j u_{-1}^{j+1}) \\ &= \kappa^{2j+1}(u_2 - u_{-1})(u_2u_{-1})^j \\ &\in \mathfrak{p}^{2j+1}\backslash\mathfrak{p}^{2j+2}\end{aligned}$$

where the last step is true because both $(u_2 - u_{-1})$ and clearly $(u_2u_{-1})^j$ are units. Therefore $(\kappa^{j+1} \wedge \kappa^j)S_2$ has κ-weight $2j+2$. □

Similarly, we have

LEMMA 2.3. *The value* $(\kappa^{j+2} \wedge \kappa^j)S_2$ *has* κ*-weight* $2j+3$, $j \geq 0$.

PROOF. As above, using the fact that $(u_2^2 - u_{-1}^2)$ is a unit in $\mathcal{O}$. □

Just to simplify notation, we write T_1 for the map $\kappa^{-1}S_2$ (Note: This is not precisely the map T_1 in [**5**, p.162]. It differs by a unit multiple).

COROLLARY 2.4. *The value* $(\kappa^{j+1} \wedge \kappa^j)T_1$ *has* κ*-weight* $2j+1$ *and* $(\kappa^{j+2} \wedge \kappa^j)T_1$ *has* κ*-weight* $2j+2$, $j \geq 0$.

In order to define another homomorphism, we need more detailed information about $\mathcal{O} \wedge \mathcal{O}$. Let $\mathbb{Z}_5$ denote the 5-adic integers. Then we can view $\mathcal{O}$ as a free $\mathbb{Z}_5$-module of rank 4 generated by 1, θ, θ^2 and θ^3. With this view, it is clear that $\mathcal{O} \wedge \mathcal{O}$ is a free $\mathbb{Z}_5$-module of rank 6 generated by

(B1) $\theta \wedge 1,\ \theta^2 \wedge 1,\ \theta^2 \wedge \theta,\ \theta^3 \wedge 1,\ \theta^3 \wedge \theta,\ \theta^3 \wedge \theta^2$

On the other hand, by Proposition 8.3.5 of [**5**], $\mathcal{O} \wedge \mathcal{O}$ is the direct sum of a free $\mathbb{Z}_5C_5$-module of rank 1 generated by $\kappa \wedge 1$ and a free $\mathbb{Z}_5$-module generated by an element z satisfying certain conditions. In the case of $p = 5$, the element $z = \theta \wedge 1 + \theta^3 \wedge 1 + \theta^3 \wedge \theta^2$ meets the conditions. Therefore $\mathcal{O} \wedge \mathcal{O}$ is also generated over $\mathbb{Z}_5$ by

(B2) $(\kappa \wedge 1),\ (\kappa \wedge 1)\theta,\ (\kappa \wedge 1)\theta^2,\ (\kappa \wedge 1)\theta^3,\ (\kappa \wedge 1)\theta^4,\ z$

In order to convert from one basis to another, we can expand each element in B2 in terms of B1. For instance,

$$\kappa \wedge 1 = (\theta - 1) \wedge 1 = (\theta \wedge 1) - (1 \wedge 1) = \theta \wedge 1$$

and, using the diagonal action of C_5 on $\mathcal{O} \wedge \mathcal{O}$

$$\begin{aligned}(\kappa \wedge 1)\theta^3 &= (\kappa\theta^3) \wedge (\theta^3) \\ &= (\theta^4 - \theta^3) \wedge \theta^3 = \theta^4 \wedge \theta^3 \\ &= ((-\theta^3 - \theta^2 - \theta - 1) \wedge \theta^3 \\ &= -(\theta^2 \wedge \theta^3) - (\theta \wedge \theta^3) - (1 \wedge \theta^3) \\ &= (\theta^3 \wedge 1) + (\theta^3 \wedge \theta) + (\theta^3 \wedge \theta^2)\end{aligned}$$

and so on. This produces the translation matrix W

	$\theta \wedge 1$	$\theta^2 \wedge 1$	$\theta^2 \wedge \theta$	$\theta^3 \wedge 1$	$\theta^3 \wedge \theta$	$\theta^3 \wedge \theta^2$
$\kappa \wedge 1$	1	0	0	0	0	0
$(\kappa \wedge 1)\theta$	0	0	1	0	0	0
$(\kappa \wedge 1)\theta^2$	0	0	0	0	0	1
$(\kappa \wedge 1)\theta^3$	0	0	0	1	1	1
$(\kappa \wedge 1)\theta^4$	1	1	0	1	0	0
z	1	0	0	1	0	1

We can now define a homomorphism $T^* \in \mathrm{Hom}_{C_p}(\wedge^2\mathcal{O},\ \mathcal{O}/\mathfrak{p}^{n-m})$ where $n > m \geq 4$ are integers which will be specified later. For now, $n - m$ can be viewed as an arbitrary positive integer. Our map will be defined using the generators B2 above. In particular,

$$(\kappa \wedge 1)T^* = \mathfrak{p}^{n-m}, \qquad zT^* = \kappa^{n-m-1} + \mathfrak{p}^{n-m}.$$

LEMMA 2.5. *The values* $(\kappa^2 \wedge 1)T^*$, $(\kappa^2 \wedge \kappa)T^*$, *and* $(\kappa^3 \wedge 1)T^*$ *all have* κ*-weight* $n - m$ *and* $(\kappa^i \wedge \kappa^j)T^* = \mathfrak{p}^{n-m}$ *for all other values of* $i > j \geq 0$.

PROOF. The value of $(x)T^*$ is determined by the coefficient of z in x relative to the basis B2 above. If that coefficient is a multiple of 5, then, since $(5) = \mathfrak{p}^4 = (\kappa^4)$, the κ-weight of $(x)T^*$ will be at least 4 + the κ-weight of $(z)T^*$. That is $4+n-m$ and so, in the quotient module, $(x)T^* = \mathfrak{p}^{n-m}$. However, if i or j is at least 4, we can factor out a scalar value of 5 showing that the coefficient of z must be a multiple of 5. Hence, $(\kappa^i \wedge \kappa^j)T^* = \mathfrak{p}^{n-m}$ for $i > j \geq 4$. To finish the proof, it suffices to compute the z component of $\kappa^i \wedge \kappa^j$ for $3 \geq i > j \geq 0$. In each case, we can expand $\kappa^i \wedge \kappa^j$ into a linear combination of basis B1 and then use the translation matrix W to switch to basis B2. In that way we find

Coefficient of z

$\wedge$	1	κ	κ^2
κ	0		
κ^2	-1	1	
κ^3	4	-5	5

In particular, we note that the κ-weights of the remaining values of T^* are as claimed. □

3. Groups

We recall some standard notation. Let G be a maximal class p-group of order p^n and $\gamma_i(G)$ denote the terms of the lower central series. Let $P_i = P_i(G) = \gamma_i(G)$ for $2 \leq i \leq n$ and let $P_1 = P_1(G)$ be the centralizer in G of P_2/P_4, and $P_0 = G$. Then the P_i form a chief series of G.

Let s and s_1 denote elements of G with $s \in G \backslash P_1$ and $s_1 \in P_1 \backslash P_2$ and define $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n$. If G has positive degree of commutativity, then Lemma 3.2.4 of [5] says that $P_i = \langle s_i \rangle P_{i+1}$, for $1 \leq i \leq n$. In this case it follows that every element of G has a unique representation of the form $s^{e_0} s_1^{e_1} s_2^{e_2} \cdots s_{n-1}^{e_{n-1}}$ where $0 \leq e_i < p$.

Following [5, p.157], let G be a 5-group of maximal class of order 5^n with positive degree of commutativity. Suppose that P_1 is class 2 and let m be such that $P_1' = P_m$. Then P_1/P_m and P_m are abelian. By Lemma 8.2.1 of [5], we have $\mathcal{O}$-module isomorphisms $f_G : \mathcal{O}/\mathfrak{p}^{m-1} \to P_1/P_m$ and $g_G : \mathcal{O}/\mathfrak{p}^{n-m} \to P_m$ given by

$$(\mathfrak{p}^{m-1} + a_0 + a_1\kappa + \cdots + a_{m-2}\kappa^{m-2})f_G = P_m s_1^{a_0} s_2^{a_1} \ldots s_{m-1}^{a_{m-2}}$$

$$(\mathfrak{p}^{n-m} + a_0 + a_1\kappa + \cdots + a_{n-m-1}\kappa^{n-m-1})g_G = s_m^{a_0} s_{m+1}^{a_1} \ldots s_{n-1}^{a_{n-m-1}}.$$

Then commutation in P_1 induces a homomorphism η_G from $\wedge^2(P_1/P_m) \to P_m$. Define

$$\alpha_G = (f_G \wedge f_G)\eta_G g_G^{-1} : \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1} \to \mathcal{O}/\mathfrak{p}^{n-m}.$$

Note that α_G is built out of commutation and, in particular, if $\zeta = (\kappa^i \wedge \kappa^j)\alpha_G$, then ζg_G is just the commutator $[s_{i+1}, s_{j+1}]$.

The next theorem provides some details about this homomorphism α_G. In order to describe α_G it is useful to note that the homomorphisms T_1 and T^* map from $\mathcal{O} \wedge \mathcal{O}$ to $\mathcal{O}/\mathfrak{p}^{n-m}$. Now, $\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1} \cong (\mathcal{O} \wedge \mathcal{O})/I$ for some C_5-submodule I. In [4, Section 7] it is shown that this I is in the kernel of each of T_1 and T^* and so each induces a homomorphism from $\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}$ to $\mathcal{O}/\mathfrak{p}^{n-m}$.

THEOREM 3.1. *Let G be a group of maximal class of order 5^n with $P_1' = P_m$ central in P_1 where $n \geq m \geq 4$. We assume G has positive degree of commutativity (this only rules out a few groups of order 5^6).*

If P_1 is not abelian, then G corresponds to a homomorphism α_G induced by $aT_1 + bT^*$ where $a \in \mathcal{O}$, $0 \leq b \leq 4$, and if $a \in \mathfrak{p}$, then $n = m+1$ and $b \neq 0$. Also one of the following holds:

(1) $m \equiv 1 \bmod 4$ *and* $2m \geq n+1$,
(2) $m > 4$, $m \not\equiv 1 \bmod 4$ *and* $2m \geq n+2$,
(3) $m = 4, n = 7$ *and* $b \equiv a \bmod p$,
(4) $m = 4, n = 5$ or 6 *and* $b = 0$.

PROOF. This is part of Theorem 7.6 of [4]. □

We now wish to compute the pattern of commutators in P_1 for some special cases of α_G.

REMARK 3.2. Any value of κ-weight k is mapped by g_G to an element of the form

$$s_{m+k-1}^{a_{m+k-1}} s_{m+k}^{a_{m+k}} \cdots s_{n-1}^{a_{n-m-1}}$$

where a_{m+k-1} is not zero.

LEMMA 3.3. *If G is a group with α_G induced by T_1, then $|P_i'| = p^2|P_{i+1}'|$ unless $|P_{i+1}'| = 1$. If $|P_{i+1}'| = 1$ then $|P_i'| \leq p^2$.*

PROOF. Fix r and consider P_r'. This subgroup is generated by the g_G-images of $\{(\kappa^i \wedge \kappa^j)T_1\}$ where $m-1 > i > j \geq r-1$. By Corollary 2.4 these T_1 values include items of κ-weight $2r-1, 2r, 2r+1, \ldots, 2m-5$. We want to know that $2m-5 \geq n-m$ or, equivalently, $3m \geq n+5$. By Theorem 3.1 $2m \geq n+1$ and since $m \geq 4$, we have $3m \geq n+5$. So, we can say that the T_1 values above include items of κ-weight $2r-1, 2r, 2r+1, \ldots, n-m$.

Based on these κ-weights, the g_G-images of these values will include elements of G with leading terms $s_{m+2r-2}, s_{m+2r-1}, \ldots, s_{n-1}$. It follows that $P_r' = P_{m+2r-2}(G)$. From this formula, the stated conditions on $|P_i'|$ are immediate. □

LEMMA 3.4. *If G is a group with α_G induced by T^*, then $|P_1'| = |P_2'| = 5$, and $|P_i'| = 1$ for $i \geq 3$.*

PROOF. By Corollary 2.5, the only non-trivial values of $(\kappa^i \wedge \kappa^j)T^*$ have κ-weight $n-m$ and so $P_1' = P_2' = P_{n-1}(G)$. □

Combining these, we have

THEOREM 3.5. *The possible values of the sequence $|P_1'|, |P_2'|, \ldots$ for a maximal class 5-group with P_1 of nilpotence class 2 are:*

$p^{2k-1}, p^{2k-3}, \ldots, p, 1, 1, \ldots$
$p^{2k}, p^{2k-2}, \ldots, p^2, 1, 1, \ldots$
$p^{2k}, p^{2k-2}, \ldots, p^2, p, 1, 1, \ldots$ *for* $k \geq 1$

or

$p, p, 1, 1, \ldots$

PROOF. We first note that the computations above often assume that G has positive degree of commutativity. This is guaranteed if $|G| > 5^6$. Using the Small-Groups database [1], we check the properties of small maximal class 5-groups with P_1 having class 2. The 6 groups of order 5^5 and 25 groups of order 5^6 have sequences $|P_1'|, |P_2'|, \ldots$ equal to $(5, 1, \ldots)$, $(5, 5, 1, \ldots)$, or $(25, 1, \ldots)$.

Now we can assume that G has positive degree of commutativity and so $\alpha_G = aT_1 + bT^*$ as above.

First we consider $b = 0$. By Lemma 3.3, the desired result holds if $a = 1$ and, similary, if the κ-weight of a is 1. However, if the κ-weight of a is greater than 1 then the κ-weight of $(\kappa^i \wedge \kappa^j)aT_1$ is uniformly larger than $(\kappa^i \wedge \kappa^j)T_1$ and so the indices $|P_i' : P_{i+1}'|$ will not change unless the subgroups in question become trivial. Consequently, the sequence $|P_1'|, |P_2'|, \ldots$ will still fall into one of the patterns given, but the values will be smaller and will reach 1 sooner.

Now if $b > 0$, the addition of $(\kappa^i \wedge \kappa^j)bT^*$ will affect at most $|P_1'|$ and $|P_2'|$. Furthermore, since we are only introducing values of κ-weight $n-m$, the orders of the commutator subgroups will only be affected if they are trivial. That is, sequences of the form $25, 1, \ldots$ and $5, 1, \ldots$ will become $25, 5, 1, \ldots$ and $5, 5, 1, \ldots$ each of which are in the stated list. □

4. Character Degrees

If G is normally monomial, the sequence $|P_1'|, |P_2'|, \ldots$ is sufficient to compute the character degrees of G as follows.

LEMMA 4.1. *Let G be a normally monomial p-group of maximal class. Then* $\mathrm{cd}(G) - 1 = \{|G : P_{i+1}|, 0 \leq i < n$ *such that* $P_i' > P_{i+1}'\}$.

PROOF. This result is found in the proof of Corollary 2.6 in [**3**]. □

This allows us to classify the possible character degrees when P_1 has class 1 or 2.

THEOREM 4.2. *Let G be a normally monomial, maximal class 5-group with $P_1(G)$ at most class 2. Then* $\mathrm{cd}(G)$ *is either* $\{1, 5, 125\}$ *or the set of all powers of 5 up to some limit,* $\{1, 5, 25, \ldots, 5^k\}, k \geq 1$.

PROOF. If P_1 is abelian, then G has an abelian group of index p and so the possible character degrees are 1 and p.

Otherwise, P_1 has class 2 and we can apply Theorem 3.5 to deduce possible values for $|P_1'|, |P_2'|, \ldots$. In particular, the non-trivial orders if P_i' strictly decrease in every case except $(5, 5, 1, \ldots)$. For these strictly decreasing sequences, Lemma 4.1 implies that the character degrees of G will form a full set of powers of 5 up to some limit, $\{1, 5, 25, \ldots, 5^k\}, k \geq 1$.

On the other hand a commutator subgroup pattern of $5, 5, 1, \ldots$ implies

$$\mathrm{cd}(G) = \{1, |G : P_1|, |G : P_3|\} = \{1, 5, 125\}$$

□

Now, a result of Mann's will allow us to lift this character degree information to any normally monomial, maximal class 5-group (regardless of class of $P_1(G)$).

LEMMA 4.3. *Let G be a normally monomial p-group satisfying $|G : G'| = p^2$, and let* $\mathrm{cd}(G) = \{1, p, p^{r_3}, \ldots, p^{r_k}\}$. *If M is a maximal subgroup of G, then* $\mathrm{cd}(M)$ *consists of 1, possibly p, and the numbers $p^{r_i - 1}$.*

PROOF. This is one case of Corollary 13 of [**6**]. □

THEOREM 4.4. *Let G be a normally monomial, maximal class 5-group. Then* $\mathrm{cd}(G)$ *is either* $\{1, 5, 25, 5^4\}$*, the set* $\{1, 5, 25, \ldots, 5^k\}$ *with* $k \geq 1$ *of all powers of 5 up to some limit, or either of those two forms with degree 25 removed.*

PROOF. Let G be any normally monomial maximal class 5-group, and let M be a maximal subgroup G not equal to P_1. Then, by [**6**], M is normally monomial and maximal class. Furthermore, by Corollary 3.4.12 of [**5**] (with $p = 5$), $P_1(M) = P_2(G)$ has class at most 2. Thus, by the previous section, $\mathrm{cd}(M)$ is constrained. Now, the preceding lemma shows that $\mathrm{cd}(G)$ is closely determined by $\mathrm{cd}(M)$ and so we deduce that $\mathrm{cd}(G)$ must be one the forms listed. □

5. Future Directions

As mentioned in the Introduction, I only know of maximal class 5-groups which have character degrees $\{1, 5, 5^3\}$ or $\{1, 5, 25, \ldots, 5^k\}$. Thus the current result, while nice, is probably not the end of the story, even for 5-groups.

A natural question is to ask what happens for $p = 7, 11, \ldots$. It seems likely that the character degrees of maximal class 7-groups will have all of the 5-group patterns (i.e. $\{1, 7, 7^3\}$ and $\{1, 7, 49, \ldots, 7^k\}$) and it appears from very preliminary computations that some other patterns of powers of 7 show up as well. I conjecture that (as with $p = 5$) there are sets of powers of 7 containing 1 and 7 that don't appear as character degree sets of any maximal class 7-group.

There are a few difficulties in applying the techniques of this paper to $p = 7$ (and higher). For $p = 7$, the homomorphisms α_G which arise are linear combinations of T_1, T^* and another map T_2. Linear combinations of T_1 and T_2 seem to have more opportunities for interaction which will probably make the case analysis harder. Similarly, the structure of $\mathcal{O} \wedge \mathcal{O}$ is more complicated. It remains true that maximal class 7-groups have derived length at most 2, but, by $p = 11$, groups of derived length 3 and more begin to appear.

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