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Digraphs with Isomorphic Underlying and Domination Graphs: Pairs of Paths

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Abstract

A domination graph of a digraph $D$, $dom(D)$, is created using the vertex set of $D$ and edge $uv \in E(dom(D))$ whenever $(u, z) \in A(D)$ or $(v, z) \in A(D)$ for any other vertex $z \in V(D)$. Here, we consider directed graphs whose underlying graphs are isomorphic to their domination graphs. Specifically, digraphs are completely characterized where $UG^c(D)$ is the union of two disjoint paths.

1 Introduction

Domination graphs were first introduced by Merz, Lundgren, Reid and Fisher [10] to describe the structure of the domination graphs and competition graphs of tournaments. Since that time, further refinements have been made in the work on tournaments, including that done by Cho, Doherty, Kim and Lundgren ([1], [2]) and Merz et al. ([6], [7], [8], [9], [10], [11]). However, the characterization of the structure of the domination graph of arbitrary digraphs has remained elusive. The authors have added to the knowledge in this area by characterizing digraphs $D$ where the underlying graph of $D$ is equal to its domination graph [3], and have characterized some digraphs where the graphs are isomorphic ([4], [5]). We add to that body of knowledge in this paper by characterizing digraphs whose underlying and domination graphs are isomorphic, $UG(D) \cong dom(D)$, and $UG^c(D)$ is the graph of two disjoint paths.

Let $D$ be a directed graph, or digraph, with nonempty vertex set $V(D)$ and arc set $A(D)$. If $(u, v) \in A(D)$, then $u$ is said to dominate $v$. Further, $u$ is the origin of the arc $(u, v)$, and $v$ is the terminating vertex. When for every other vertex $z$ in $V(D)$, either $(u, z)$ or $(v, z)$ is an arc in $D$, then $u$ and $v$ form a
The domination graph of $D$, $\text{dom}(D)$, is an undirected graph with the vertex set $V(D)$, where there is an edge between every dominating pair. A digraph $D$ is considered a biorientation of a graph $G$ if for every edge $uv \in E(G)$, either $(u, v)$ or $(v, u)$ or both are arcs in $D$, and $D$ contains no other arcs. The underlying graph of $D$, $\text{UG}(D)$, is the graph for which $D$ is a biorientation. If for edge $uv$ in $G$, only one of edges $(u, v)$ or $(v, u)$ is in $D$, then the arc is called an orientation of edge $uv$. When all edges of $G$ are bidirected edges in $D$, then $D$ is a complete biorientation of $G$, also known as a symmetric digraph. Although bidirected edges are allowed in $D$, they are considered arcs in $D$.

When the underlying graph of $D$ is isomorphic to the domination graph of $D$, it is its nature to have many edges. Thus, in most cases, it is easier to obtain the results regarding $\text{UG}(D)$ and $\text{dom}(D)$ by observing patterns in $\text{UG}^c(D)$ and $\text{dom}^c(D)$, which are sparse graphs. To relate the results obtained from the complements to $\text{UG}(D)$ and $\text{dom}(D)$, we use the concepts of the union and the join of graphs and digraphs. The union of two graphs or digraphs is the graph or digraph formed by the union of their vertices as well as their sets of edges or arcs. The join of two graphs $G$ and $H$, $G + H$, is the graph that consists of $G \cup H$ and all edges joining the vertices in $G$ with the vertices in $H$. We extend this definition to directed graphs as follows. The join of $D_1$ and $D_2$ consists of $D_1 \cup D_2$ together with all bidirectional edges between every vertex of $D_1$ and every vertex of $D_2$.

We know that the structure of $\text{UG}(D)$ is limited to a small number of constructs. It can be summed up by the following three results.

**Theorem 1.1** \cite{4} If $D_1, \ldots, D_k$ are directed graphs such that $\text{UG}(D_i) \cong \text{dom}(D_i)$ for $i = 1, \ldots, k$ and $D = D_1 + D_2 + \ldots + D_k$, then $\text{UG}(D) \cong \text{dom}(D)$. Also

1. $\text{UG}(D) = \bigcup_{i=1}^k \text{UG}(D_i)$
2. $\text{dom}(D) = \bigcup_{i=1}^k \text{dom}(D_i)$
3. $\text{UG}^c(D) = \bigcup_{i=1}^k \text{UG}^c(D_i)$
4. $\text{dom}^c(D) = \bigcup_{i=1}^k \text{dom}^c(D_i)$

**Theorem 1.2** \cite{4} If $\text{UG}(D)$ is isomorphic to $\text{dom}(D)$, then $\text{UG}^c(D)$ is comprised of one or more connected components, each either a complete graph, a path, or a cycle.

**Corollary 1.3** \cite{4} If $\text{UG}(D)$ is isomorphic to $\text{dom}(D)$, then $D$ is the join of $D_1, D_2, \ldots, D_k$, where $\text{UG}(D_i)$ is isomorphic to an independent set, the complement of a path, or the complement of a cycle.

Theorem 1.2 gives three basic components that comprise the complement of the underlying graph in which we are interested. The structure of $D$ and $\text{UG}(D)$ where $\text{UG}^c(D)$ is one component has been completely characterized \cite{4}, as has the case where $P_1, P_2$ and $C_4$ are the components \cite{5}. In this paper, we further the research by characterizing the underlying graphs and the directed graphs where $\text{UG}(D) \cong \text{dom}(D)$, and $\text{UG}^c(D) = P_1 \cup P_2$. In the next section, we determine for what values of $j$ $\text{UG}^c(D) = P_i \cup P_j$ exists for the special cases of $i = 1, 2$. We then use the information from Section 2 to formulate the characterizations of the digraphs $D$ that can be formed for the associated underlying graphs. Finally, we conclude the characterizations of $\text{UG}^c(D)$ and $D$ where $i, j \geq 2$ in Sections 4 and 5. Some of the proofs required are quite long and interrupt the flow of information, so have been placed in their own section at the end of the paper.

**2 Structure of $\text{UG}^c(D) = P_i \cup P_j$, $i = 1, 2$**

Of immediate consequence when determining the structure of $\text{UG}^c(D)$ for any $i$ and $j$ is the edges that are formed in $\text{dom}^c(D)$ regardless of the structure of $D$. The following lemma lists the paths that are always part of $\text{dom}^c(D)$ when $P_n$ for $n \geq 3$ is a component of $\text{UG}^c(D)$. These paths are used extensively in this paper, and will be referred to as the generated subpaths in $\text{dom}^c(D)$.

**Lemma 2.1** \cite{4} If $\text{UG}^c(D) = P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$, then

1. if $n$ is odd, $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $\text{dom}^c(D)$, and
2. if $n$ is even, $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $\text{dom}^c(D)$.

Further, we know that a biorientation of $\text{UG}(D)$ exists for each of the $P_n$, $n \geq 3$, where $\text{UG}(D) \cong \text{dom}(D)$. This is stated in the following lemma.

**Lemma 2.2** \cite{4} Let $D$ be a directed graph on $n \geq 3$ vertices and $\text{UG}^c(D) = P_n = x_1, \ldots, x_n$. Then $\text{dom}^c(D) \cong P_n$ if and only if for every edge $uv \in E(\text{UG}(D))$, $(u, v)$ and $(v, u)$ are arcs in $D$ except for the following:

1. if $n$ is odd, exactly one of the following is an orientation of the associated edge(s) in $\text{UG}(D)$:
   
   (a) $(x_1, x_n)$,
   (b) $(x_n, x_1)$,
   (c) $(x_1, x_n)$ and $(x_n, x_{n-3})$, or
   (d) $(x_n, x_1)$ and $(x_1, x_4)$,

   2. if $n$ is even, exactly one of the following is an orientation of the associated edge(s) in $\text{UG}(D)$:

   (a) $(x_1, x_{n-1})$,
   (b) $(x_n, x_2)$,
   (c) $(x_1, x_{n-1})$ and $(x_n, x_2)$,
(d) \((x_n, x_3)\) and \((x_1, x_4)\), or
(e) \((x_1, x_n-1)\) and \((x_n, x_{n-3})\).

Of particular note, the oriented edges \((x_n, x_{n-3})\) and \((x_1, x_4)\) of the preceding lemma form edges in \(\text{dom}^c(D)\) that are in the generated subpaths of Lemma 2.1. These serve a special purpose when we characterize \(D\), so the following two corollaries are listed here for later use. The first follows from construction of \(\text{dom}^c(D)\).

**Corollary 2.3** If \(UG^c(D) = P_n, n \geq 4\), oriented edges \((x_1, x_4)\) and \((x_n, x_{n-3})\) produce edges \(x_2x_4\) and \(x_{n-1}x_{n-3}\) in \(\text{dom}^c(D)\).

Further, this guarantees that when we use oriented edges \((x_n, x_{n-3})\) and \((x_1, x_4)\), there will be no new edges appearing in \(\text{dom}^c(D)\).

**Corollary 2.4** If \(UG^c(D) = P_n, n \geq 4\), oriented edges \((x_1, x_4)\) and \((x_n, x_{n-3})\) create no additional edges in \(\text{dom}^c(D)\).

To show that \(UG^c(D) = P_1 \cup P_2\) can exist for all \(j \geq 1\) such that \(UG(D) \cong \text{dom}(D)\), we need to show that if the component \(P_j\) is added to the graph \(UG^c(D) = P_1\), an underlying graph will be created where it is possible to still create a digraph \(D\) preserving isomorphism. We can do that using the orientations given in Lemma 2.2.

**Theorem 2.5** Let \(UG^c(D) = P_1 \cup P_2\). For all \(j \geq 1\), there exists a biarcivation of the edges of \(UG^c(D)\) such that \(UG(D) \cong \text{dom}(D)\).

**Proof.** Let \(j = 1\). Then \(UG(D)\) equals the edge \(uv\). The vertices \(u\) and \(v\) dominate for either orientation of the edge \(uv\) or the bidirection of the edge. Thus, \(UG(D) \cong \text{dom}(D)\).

Let \(j = 2\). Then \(UG(D) = w_1 \cup w_2\). The orientation \((u, v_1) \cup (u, v_2)\) produces edges \(w_1\) and \(w_2\) in \(\text{dom}(D)\). Thus, \(UG(D) \cong \text{dom}(D)\).

Let \(j \geq 3\). If \(D_1 = P_1\) and \(D_2 = P_2\), where \(UG^2(D) = D_1 \cup D_2\), then by Theorem 1.1, \(UG(D) \cong \text{dom}(D)\) where \(D = D_1 \cup D_2\).

Now we turn our attention to characterizing the \(j\) for which \(UG(D) \cong \text{dom}(D)\) and \(UG^c(D) = P_1 \cup P_2\). To do so, there must be more understanding of the orientation of the edges to form \(D\) and the affect this has on \(\text{dom}^c(D)\). We work with \(\text{dom}^c(D)\) because when \(UG(D) \cong \text{dom}(D)\), \(UG^c(D) \cong \text{dom}^c(D)\), and it is easier to work with the fewer edges in the complements.

Given any edge \(uv\) in \(\text{dom}^c(D)\), we know that vertices \(u\) and \(v\) cannot form a dominating pair in \(D\). Therefore, there must be at least one vertex \(z\) in \(D\) such that neither \((u, z)\) nor \((v, z)\) is an arc. We will call \(z\) a source of edge \(uv\) in \(\text{dom}^c(D)\). Note that an edge in \(\text{dom}^c(D)\) may have multiple sources.

The next few results eliminate certain vertices as candidates for sources, and restrict the number of edges for which a vertex may be a source. In our construction of a digraph where \(UG^c(D)\) is the union of two paths, it is natural to ask whether it is possible for a vertex to be the source of more than one edge in \(\text{dom}^c(D)\).

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**Lemma 2.6** [5] If \(UG(D) \cong \text{dom}(D)\) and \(y\) is the source of two or more edges in \(\text{dom}^c(D)\), then the set of vertices which do not dominate \(y\) is contained in a component isomorphic to \(K_r, r \geq 3\) in \(UG^c(D)\).

Since we have no components isomorphic to \(K_r, r \geq 3\) in \(UG^c(D)\), we have no vertices that are the source of more than one edge in \(\text{dom}^c(D)\). There are two ways in which a vertex \(z\) may be a source of edge \(uv\) in \(\text{dom}^c(D)\). The first is if \(z\) is not adjacent to \(u\) and \(v\), then \(z\) is not adjacent to \(u\) or \(v\) if \(z\) is not adjacent to both \(u\) and \(v\), or both if \(z\) is adjacent to both \(u\) and \(v\). We can obtain the list of vertices that are candidates for becoming the origin of an oriented edge. The following lemma is used as the foundation for the choices.

**Lemma 2.7** [4] Let \(D\) be a digraph on \(n\) vertices, and \((u, v)\) in \(D\) be the orientation of edge \(uv\) in \(UG^c(D)\), where \(\deg(u) = k\) in \(UG^c(D)\). If \(k < n - 2\), then \(K_3\) is a subgraph of \(\text{dom}^c(D)\).

The preceding lemma thus leads to the following set of vertices that may serve as the origin for any oriented edge in \(D\) when \(UG(D) \cong \text{dom}(D)\) and \(UG^c(D)\) is comprised of disjoint paths.

**Lemma 2.8** Let \(D\) be any digraph such that \(UG(D) \cong \text{dom}(D)\) and \(UG^c(D)\) is comprised of components \(\bigcup_{i=1}^{k} P_{n_i}\), where \(P_{n_i} = x_{i1}, x_{i2}, \ldots, x_{in_i}\) and \(n_i \geq 1\) is the number of vertices for path \(P_{n_i}\). If \(UG(D) \cong \text{dom}(D)\) and \((u, v)\) in \(D\) is an orientation of edge \(uv\) in \(UG^c(D)\), then \(u = x_{i1}\) or \(u = x_{i,n_i}\) for some \(j, 1 \leq j \leq k\).

**Proof.** Consider \(UG^c(D) = \bigcup_{i=1}^{k} P_{n_i}\), where \(P_{n_i} = x_{i1}, x_{i2}, \ldots, x_{in_i}\) and \(n_i \geq 1\), and \((u, v)\) is an orientation of edge \(uv\). Let \(n = \sum_{i=1}^{k} n_i\). According to Lemma 2.7, if \(\deg(u) < n - 2\) in \(UG^c(D)\), then \(K_3\) is a subgraph of \(\text{dom}^c(D)\). Thus, \(\deg(u) \geq n - 2\). This indicates that in \(UG^c(D)\), \(\deg(u) \leq 1\). So, \(u\) must be \(K_1\), or the end vertex of a path. Therefore, we obtain the list of vertices, which are the first and last vertices of each \(P_{n_i}\).

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**Lemma 2.9** Let \(D\) be any digraph such that \(UG(D) \cong \text{dom}(D)\) and \(UG^c(D)\) is comprised of components \(\bigcup_{i=1}^{k} P_{n_i}\), where \(P_{n_i} = x_{i1}, x_{i2}, \ldots, x_{in_i}\) and \(n_i \geq 2\) is the number of vertices for path \(P_{n_i}\). If \((z, u)\) in \(D\) is an orientation of edge \(uz\) in \(UG^c(D)\), then there is no vertex \(v\) such that \((z, v)\) in \(D\) is an orientation of edge \(uv\) in \(UG^c(D)\).
Proof. Suppose that there are orientations \((z, u)\) and \((z, v)\) in \(D\). By Lemma 2.8, \(z\) must be one of the end vertices of the path, and there exists a vertex \(z_k\) that is not adjacent to \(z\) in \(UG(D)\). So, \(z_k\), \(u\) and \(v\) do not dominate \(z\) and form \(K_3\) in \(dom^c(D)\), contradicting \(UG(D) \cong dom(D)\). Thus, \((z, u)\) or \((z, v)\) may be an orientation of an edge in \(D\), but not both.

Now we turn our attention to the structure of the paths themselves in \(dom^c(D)\). The generated subpaths in \(dom^c(D)\) are only in constructions for \(P_2\) when \(j \geq 3\). However Lemma 2.1 does give an indication of the length of the paths formed automatically in \(dom^c(D)\). As \(j\) becomes larger, the generated subpaths in \(dom^c(D)\) on the vertices \(v_1,\ldots,v_j\) become longer than \(P_2\). Thus, it is necessary to know if \(P_2 = u_1u_2\) in \(UG^c(D)\) can also form \(P_2 = u_1v_2\) in \(dom^c(D)\). If not, there are only a few possible values for \(j\) so that \(UG^c(D) = P_2 \cup P_3\) can yield an isomorphic \(dom^c(D)\). The following lemma states that \(P_2 = u_1u_2\) in \(UG^c(D)\) is not possible.

**Lemma 2.10** Let \(UG(D) \cong dom(D)\), and \(UG^c(D) = P_2 \cup P_j\) for \(j \geq 3\), where \(P_2 = u_1u_2\) and \(P_j = v_1,\ldots,v_j\). Then \(u_1u_2\) is not an edge in \(dom^c(D)\).

**Proof.** Suppose that \(u_1u_2\) is an edge in \(dom^c(D)\). Then some vertex \(z\) must be a source of that edge. Vertices \(u_1\) and \(v_2\) cannot be the source of an edge with which they are incident. Thus, \(z = v_k\) for some \(k = 1,\ldots,j\). Since \(v_k\) is adjacent to both \(u_1\) and \(u_2\), the oriented edges \((v_k, u_1)\) and \((v_k, u_2)\) must both be in \(D\). But Lemma 2.9 states that this cannot be. Therefore, \(u_1u_2\) is not an edge in \(dom^c(D)\).

**Corollary 2.11** Let \(UG(D) \cong dom(D)\), and \(UG^c(D) = P_2 \cup P_j\) for \(j \geq 3\), where \(P_2 = u_1u_2\) and \(P_j = v_1,\ldots,v_j\). Then \(P_j\) in \(dom^c(D)\) is either equal to \(u_1v_k\) for some \(i = 1, 2\) and some \(k = 1,\ldots,j\) or \(v_iv_k\) for some \(i = 1,\ldots,j\) and \(k = 1,\ldots,j\).

Now we can formulate the structure of \(UG^c(D)\) given the preceding results. When it is shown that \(UG^c(D) \cong dom^c(D)\), we generally skip directly to the consequence of \(UG(D) \cong dom(D)\). Figure 1 shows the construction for \(j = 4\) given in the proof for Theorem 2.12. Bidirectional edges are not shown. The dashed lines represent the edges in \(UG^c(D)\), so are not bidirected edges in \(D\).

In the figure, \(P_2 = v_2v_4\) in \(dom^c(D)\).

**Theorem 2.12** Let \(UG^c = P_2 \cup P_j\). There exists a biorientation \(D\) of the edges of \(UG(D)\) such that \(UG(D) \cong dom(D)\) if and only if \(j = 1, 2, 3, 4, 5\).

**Proof.** (\(\implies\)) Theorem 2.5 shows the case where \(j = 1\). For \(j \geq 2\), according to Corollary 2.11, we must construct \(P_j\) in \(dom^c(D)\) using \(u_kv_k\) or \(v_kv_k\). First consider \(P_2 = u_kv_k\) in \(dom^c(D)\). Here, \(v_k\) must be the generated subpath \(P_1\) so that \(u_kv_k = P_2\) in \(dom^c(D)\). Therefore, \(j \leq 3\). Next consider \(P_2 = v_kv_k\) in \(dom^c(D)\). The edge \(v_kv_k\) must be the generated subpath \(P_2\), so Lemma 2.1 gives us \(j \leq 5\).

(\(\Longleftarrow\)) The case where \(j = 1\) is shown in Theorem 2.5.

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![Figure 1: A digraph D where UG^c(D) = P_2 \cup P_4 and the associated graph dom^c(D). Dashed lines represent UG^c(D), and bioriented edges of D are omitted.](image-url)
an edge in \( \text{dom}^c(D) \). Say that \( uv_1 \) is an edge. Then there is a source vertex \( z \) in \( D \) such that neither \( u \) nor \( v_1 \) dominates \( z \). By Lemma 2.9, \( z \) cannot be adjacent to both \( u \) and \( v_1 \). Vertex \( u \) is adjacent to all other vertices in \( \text{UG}(D) \), so \( z = v_2 \). This implies that \( (v_2, u) \) must be an oriented edge so that \( v_2 \) and \( u \) do not dominate \( v_1 \). However, \( v_2 \) cannot be the origin of an oriented edge according to Lemma 2.8. Thus, \( uv_1 \) is not an edge in \( \text{dom}^c(D) \). With a similar argument, we see that \( uv_3 \) is not an edge in \( \text{dom}^c(D) \). Thus, \( u = P_1 \) in \( \text{dom}^c(D) \) is the only possibility. It can be realized by applying the assignment of oriented edges associated with \( P_3 \) outlined in Lemma 2.2, and biorienting the edges \( uv_1 \) for \( i = 1, 2, 3 \).

If \( j \geq 4 \), then \( P_j \) is not a generated subpath in \( \text{dom}^c(D) \), so the only possibility is vertex \( u \). The graph \( \text{dom}^c(D) \) that is isomorphic to \( \text{UG}(D) \) can be realized by applying the assignment of oriented edges associated with \( P_3 \) outlined in Lemma 2.2, and biorienting the edges \( uv_i \) for \( i = 1, \ldots, j \).

This leads to the following characterization of digraphs where \( \text{UG}(D) \cong \text{dom}(D) \) and \( \text{UG}^c(D) = P_1 \cup P_2 \).

**Theorem 3.2** Let \( \text{UG}^c(D) = P_1 \cup P_2 \), where \( P_1 = u \) and \( P_2 = v_1, \ldots, v_j \). \( \text{UG}(D) \cong \text{dom}(D) \) if and only if \( D \) is of the form:

1. If \( j = 1 \), then \( D \) is an orientation of the edge \( uv_1 \) or the biorientation of that edge.
2. If \( j = 2 \), then \( (u, u) \) is an orientation for \( i = 1 \) or \( 2 \), and \( (u, v, u) \) is not an orientation for \( i = 1 \) or \( 2 \) and \( i' \) the remaining value, or \( (u, v_1) \) and \( (u, v_2) \) are orientations.
3. If \( j \geq 3 \) and \( v_p, v_q \) is the edge in \( \text{dom}^c(D) \) connecting the generated subpaths, then \( D \) is the digraph where all edges of \( \text{UG}(D) \) are bioriented in \( D \) except for one of the following:

   a) the only oriented edges are as described in Lemma 2.2, or
   b) \( (u, v_p) \) and \( (u, v_q) \) are orientations of edges \( uv_p \) and \( uv_q \) respectively, or
   c) the edges are oriented as described in Lemma 2.2, and \( u \) is the origin of at most two oriented edges \( (u, v_k) \) and \( (u, u) \), \( k < l \), where
   i. \( u \) is the origin of only one oriented edge, \( (u, v_k) \), of edge \( uv_k \) for \( k = 1, \ldots, j \); or
   ii. \( u \) is the origin of two oriented edges where \( k = 1, \ldots, j - 2 \) and \( l = k + 2 \), or \( k = p \) and \( l = q \).

The proof of Theorem 3.2 can be found in the final section of this paper.

When \( i = 1 \), it is possible for a single vertex, namely \( u \), to be the origin of two oriented edges. However, once \( i, j \geq 2 \), that possibility is eliminated, as was outlined in Lemmas 2.8 and 2.9. Although there are similarities in \( \text{UG}^c(D) = P_3 \cup P_2 \) for \( j = 2, 3, 4, 5 \), the differences are enough that we list the results separately. First, we use the following lemma and its corollary to establish source vertices for \( \text{UG}^c(D) = u_1, u_2 \cup v_1, v_2 \). The set of vertices adjacent to a vertex \( v \) is the neighborhood of \( v \).

**Lemma 3.3** [5] If \( \text{UG}(D) \cong \text{dom}(D) \) and \( n(y) = \{ x \} \) in \( \text{UG}(D) \), then \( y \) is a source of at most one edge in \( \text{dom}^c(D) \), and this edge will be incident to \( x \).

**Corollary 3.4** If \( \text{UG}(D) \cong \text{dom}(D) \) and \( \text{UG}^c(D) = u_1, u_2 \cup v_1, v_2 \), then each vertex in \( D \) can be the source of at most one edge in \( \text{dom}^c(D) \). Furthermore, \( u_1 \) may only be the source of an edge incident with \( u_2, u_2 \) the source of an edge incident with \( u_1 \), and similarly for \( v_1 \) and \( v_2 \).

Since there are only two edges in \( \text{UG}^c(D) \), we desire only two edges in \( \text{dom}^c(D) \). Therefore, we pick only two of the vertices in \( u_1, u_2 \cup v_1, v_2 \) to be the origin of the oriented edges in \( D \). Although it might seem possible to orient all four edges in \( \text{UG}(D) \) in such a way that only two edges are formed in \( \text{dom}^c(D) \), this cannot be done. Following, all digraphs \( D \) where \( \text{UG}^c(D) = P_2 \cup P_2 \) and \( \text{UG}(D) \cong \text{dom}(D) \) are characterized.

**Theorem 3.5** Let \( \text{UG}^c(D) = P_2 \cup P_2 \). Further, let \( u \) be a vertex of one of the paths and let \( v' \) be the other vertex of that path. Let \( v \) be a vertex of the other path and \( v' \) its second vertex. \( \text{UG}(D) \cong \text{dom}(D) \) if and only if \( (u, v) \) and \( (u', v') \) are oriented edges, and all other edges of \( \text{UG}(D) \) are bioriented in \( D \).

**Proof.** (\( \Rightarrow \)) Let \( u \) be \( u_1 \) or \( u_2 \) and \( v \) be \( v_1 \) or \( v_2 \). Further, suppose that \( (u, v) \) is an oriented edge in \( D \). This creates edge \( uv' \) in \( \text{dom}^c(D) \). Since \( \text{UG}(D) \cong \text{dom}(D) \), \( \text{dom}^c(D) \) must contain only one more edge, \( uv' \). According to Corollary 3.4, only vertex \( u' \) or \( v' \) may be the source of this edge. Therefore, \( (u', v') \) or \( (v', u') \) are the possible oriented edges in \( D \) that will create the edge in \( \text{dom}^c(D) = P_2 \cup P_2 \). Since \( (u, v) \) is an oriented edge, \( (v, u) \) is not a viable choice. Thus, \( (u', v') \) must be an oriented edge in \( D \). Suppose that there are other oriented edges. No additional edges can be formed in \( \text{dom}^c(D) \). The only arc that has not been discussed earlier is \( (v', u') \), but it would bidirect edge \( uv' \), which is oriented in creating \( \text{dom}^c(D) \). Thus, there are no other oriented edges possible in \( D \). Since \( u \) is any of the four vertices in \( \text{UG}(D) \), this holds for all cases.

(\( \Leftarrow \)) If \( (u, v) \) and \( (u', v') \) are oriented edges, then \( uv \) and \( uv' \) are edges in \( \text{dom}^c(D) \). The number of vertices in each path is less than \( 3 \), so there are no generated subpaths in \( \text{dom}^c(D) \). Thus, \( \text{dom}^c(D) = P_2 \cup P_2 \), so \( \text{UG}^c(D) \cong \text{dom}(D) \), and \( \text{UG}(D) \cong \text{dom}(D) \).
Proof. If \((v_1, v_3)\) or \((v_3, v_1)\) is an oriented edge, then by Lemma 2.2, vertices \(v_1, v_2\) and \(v_3\) form \(P_3\) in \(\text{dom}^c(D)\). Since \(u_1\) and \(u_2\) cannot form \(P_3\) in \(\text{dom}^c(D)\), according to Lemma 2.10, there is no way to create \(P_3\) in \(\text{dom}^c(D)\), and \(UG^c(D) \neq \text{dom}^c(D)\). Thus, both \((v_1, v_3)\) and \((v_3, v_1)\) must be arcs in \(D\).

One characteristic that begins to appear now and will follow the constructions through all of the pairs of paths, concerns multiple sources for an edge. If more than one vertex in these digraphs can be the source of the same edge in \(\text{dom}^c(D)\), then we can use any combination of the oriented edges in \(D\) that create the edge without creating new edges in \(\text{dom}^c(D)\). However, we must be careful that each vertex is the source of at most one edge.

It is now possible to characterize all digraphs where \(UG^c(D) = P_2 \cup P_3\). Figure 2 shows a possible construct using the vertex labeling convention adopted in the theorem and its proof. The figure shows a digraph where the oriented edges are formed using part (1) of Theorem 3.7. The random choice of \(u\) and \(v\) allows the characterization of all digraphs without listing each isomorphic labeling.

![Figure 2](image)

**Figure 2:** \(D\) shows a maximum number of oriented edges when \(UG^c(D) = P_2 \cup P_3\). Vertex labeling is arbitrary.

**Theorem 3.7** Let \(UG^c(D) = P_2 \cup P_3\). Further, let \(u\) be a vertex of \(P_2\) and let \(u'\) be the other vertex. Let \(v\) be an end vertex of \(P_3\) and \(v'\) be the other end vertex. \(UG(D) \cong \text{dom}(D)\) if and only if every edge of \(UG(D)\) is bidirected in \(D\) except for the following.

1. \((u', v')\) is an oriented edge of \(UG(D)\) in \(D\), and
2. \((u, v)\) or \((v, u')\) both are oriented edges of \(UG(D)\) in \(D\), and
3. \((v, v')\) or \((v', v)\) both are arcs in \(D\) such that \(v\) and \(v'\) each are the origin of at most one oriented edge.

The proof for Theorem 3.7 can be found in the final section of this paper.

When \(UG^c(D) = P_2 \cup P_4\), we have the first instance where there are two nontrivial generated subpaths in \(\text{dom}^c(D)\). Either of these paths will be the one to form \(P_2\) in \(\text{dom}^c(D)\). Again, there is much symmetry here, so the labeling we choose gives us all possible digraphs. Figure 3 represents just one of the selections that the labeling can produce, and aids in the understanding of the proof to the theorem.

![Figure 3](image)

**Figure 3:** Example of labeling used where \(UG^c(D) = P_2 \cup P_4\).

**Theorem 3.8** Let \(UG^c(D) = P_2 \cup P_4\). Further, let \(P_2 = u, u'\) and \(P_4 = v, v_1, v_2, v'\) for arbitrary selections of end vertices \(u, u', v, v'\) in \(UG^c(D)\). \(UG(D) \cong \text{dom}(D)\) if and only if every edge of \(UG(D)\) is bidirected in \(D\) except for the following.

1. \((u', v')\) is an oriented edge of \(UG(D)\) in \(D\), and
2. \((u, v)\) or \((v, u')\) both are oriented edges of \(UG(D)\) in \(D\), and
3. \((v, v')\) or \((v', v)\) both are arcs in \(D\) such that \(v\) and \(v'\) each are the origin of at most one oriented edge.

The proof for Theorem 3.8 can be found in the final section of this paper.

![Figure 4](image)

**Figure 4:** \(D\) shows a maximum number of oriented edges when \(UG^c(D) = P_2 \cup P_3\). Dashed edges are \(UG^c(D)\), and bidirectional edges are omitted for simplicity.

For the final characterization in this section, we have \(\text{dom}^c(D)\) generated with very little choice of what vertices form \(P_2\). The generated subpaths are
Theorem 3.9 Let $UG^c(D) = P_2 \cup P_3$. Further, let $P_2 = u, u'$ and $P_3 = v, v_1, v_2, v_3, v_4, \ldots, v_i$ for arbitrary selections of end vertices $u, u', v, v_1, v_2, v_3, v_4, \ldots, v_i$ in $UG^c(D)$. If $UG(D) \cong \text{dom}(D)$ if and only if every edge of $UG(D)$ is bidirected in $D$ except for the following.

1. $(v, v')$ and $(u, u')$ are both oriented edges of $UG(D)$ in $D$, and
2. $(v, v')$ or $(u, u')$ or both or neither are oriented edges of $UG(D)$ in $D$.

Proof. Paths $v_1, v_2, v_3, v_4, \ldots, v_i$, and $v_1, v_2, v_3, v_4, \ldots, v_i$ are generated subpaths in $dom^c(D)$. Since $uu'$ cannot be an edge in $dom^c(D)$, $P_2 = v_1, v_2, v_3, v_4, \ldots, v_i$. Thus, $P_3 = u, v_1, v_2, v_3, v_4, \ldots, v_i$ in $dom^c(D)$. Edges $uv$ and $u'v'$ need to have a source in $D$. Since $v$ is an end vertex, there is no $v$ that can be used as a source of the edge for reasons explained in the proof of Theorem 3.8. Therefore, $(v, v')$ is the only oriented edge that will form $uv$ in $dom^c(D)$, so it must be in every biorientation of $UG(D)$. For similar reasons, $(u, u')$ is the only oriented edge generating $u'v'$ in $dom^c(D)$, so it must be in every biorientation of $UG(D)$, proving part 1.

Corollary 2.4 allows that we may use oriented edges $(v, v')$ and $(u, u')$ without creating new edges in $dom^c(D)$, as these oriented edges are not necessary for the production of an additional edge in $dom^c(D)$, if they are used, then they can appear in a biorientation singly or together, proving part 2. Since $u$ and $v$ in these selections of the end vertices, we obtain all possible biorientations.

The previous lemma has important consequences for $UG^c(D)$ when both $i$ and $j$ are odd. Recall that when $i$ is odd, $u_1, u_3, \ldots, u_i$ is a generated subpath in $dom^c(D)$, when we have two such paths, they can never be connected to form a larger path in $dom^c(D)$, since we cannot form edges between the end vertices.

Corollary 4.3 Let $P_i = u_1, u_3, \ldots, u_i$, and $P_j = v_1, v_3, \ldots, v_j$ be paths that are components of $UG^c(D)$ for $i, j \geq 3$. Further, let $U_1 = u_1, u_3, \ldots, u_i$, $U_2 = u_2, u_4, \ldots, u_{i-1}$, $V_1 = v_1, v_3, \ldots, v_j$, and $V_2 = v_2, v_4, \ldots, v_{j-1}$ be the generated subpaths in $dom^c(D)$, where $U_1V_1$ denotes the path $u_1, u_3, \ldots, u_i, v_1, v_3, \ldots, v_j$. If $UG(D) \cong \text{dom}(D)$, then $U_1V_1$ is not a path in $dom^c(D)$.

Now we have the information necessary to further characterize $UG^c(D)$, where we expect $dom^c(D)$ to be formed using vertices from both $V(P_i)$ and $V(P_j)$. We therefore continue the discussion of the underlying graph by expanding upon Lemma 2.8. Because only the end vertices of the paths in $UG^c(D)$ may be used as origins of oriented edges in $D$, certain edges cannot occur in $dom^c(D)$. The following lemma details the edges that will never appear in that graph.

Lemma 4.2 Let $P_i = u_1, u_3, \ldots, u_i$, and $P_j = v_1, v_3, \ldots, v_j$ be paths that are components of $UG^c(D)$ for $i, j \geq 3$. If $UG(D) \cong \text{dom}(D)$, then $u_1v_1, u_1v_3, u_3v_3, \ldots, u_iv_i$ are not edges in $dom^c(D)$. Proof. Lemma 2.8 states that only $u_1, u_i, v_i$, or $v_j$ can be the origin of an oriented edge in $D$. A vertex can also not be the source of more than one edge or an edge with which it is incident in $dom^c(D)$. Thus, to form $u_1v_i$ in $dom^c(D)$, either $u_i$ or $v_i$ must be the source. Both $u_i$ and $v_i$ are adjacent to $u_1$, so the oriented edges $(u_1, u_i)$ and $(u_1, v_i)$ cannot be in $D$ for either of the two vertices to be the source of the edge $u_1v_i$. This contradicts Lemma 2.9, so $u_1v_i$ does not have any possible source and cannot be produced in $dom^c(D)$ when $UG(D) \cong \text{dom}(D)$. Similar arguments hold for the other three edges between the end vertices of the paths.

The 4 Structure of $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$

If we were interested in seeing only what pairs of paths can comprise $UG^c(D)$ so that $UG(D) \cong \text{dom}(D)$, the answer would be simple.

Theorem 4.1 Let $UG^c(D) = P_i \cup P_j$ where $i, j \geq 3$. There exists a biorientation of the edges of $UG(D)$ such that $UG(D) \cong \text{dom}(D)$ for every value of $i, j \geq 3$.

Proof. This follows directly from Theorem 1.1 and Lemma 2.2.

However, we are interested in much more than just existence. The main goal is to characterize all digraphs where $UG^c(D) = P_i \cup P_j$ and $UG(D) \cong \text{dom}(D)$. Therefore, we must also consider the structure of $UG^c(D)$ when paths in $dom^c(D)$ are formed using vertices from both $V(P_i)$ and from $V(P_j)$.
5 Characterization of $D$ where $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$

In building digraphs as biorientations of their underlying graphs, Theorems 4.1 and 4.4 separate the characterization into two parts. The first is where $P_i$ and $P_j$ are created in $dom^c(D)$ using $V(P_i)$ and $V(P_j)$ respectively from $UG^c(D)$. The second is where generated subpaths from $V(P_i)$ and $V(P_j)$ are connected to form the paths. Now, to characterize all $D$ where $P_i$ and $P_j$ are formed in $dom^c(D)$ using Lemma 2.2, we need the following result.

**Lemma 5.1** Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be paths that are components of $UG^c(D)$ for $i, j \geq 3$. Also, let $u = u_1$ or $u_i$ and $v = v_1$ or $v_j$. Any oriented edge $(u, v)$ or $(v, u)$ for $1 \leq k \leq i$ and $1 \leq l \leq j$, creates an edge in $dom^c(D)$ between $V(P_i)$ and $V(P_j)$.

**Proof.** If $u = u_1$ or $u_i$, then $u_2v_1$ or $u_{i-1}v_i$ respectively is an edge in $dom^c(D)$. The same argument holds for $v$. ■

This result makes it clear that in the case where Lemma 2.2 is used to produce paths $P_i$ and $P_j$ in $dom^c(D)$, all edges in $UG(D)$ between the vertices of $P_i$ and $P_j$ must be bidirected.

**Theorem 5.2** Let $UG^c(D) = P_i \cup P_j$ for $i, j \geq 3$ where $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$. $UG^c(D) \cong dom^c(D)$ and $u_2v_1$ is not an edge in $dom^c(D)$ for any $1 \leq k \leq i$, $1 \leq l \leq j$, and only if the edges of $UG(P_i)$ and $UG(P_j)$ are bioriented as stated in Lemma 2.2 and all other edges are bidirected to form $D$.

**Proof.** ($\Rightarrow$) Since $UG(D) \cong dom(D)$ and $u_2v_1$ is not an edge in $dom^c(D)$, Lemma 5.1 indicates that no oriented edge may exist between $u_1, ..., u_i$ and $v_1, ..., v_j$. So all edges between $V(P_i)$ and $V(P_j)$ in $D$ are bidirected. Thus, the paths must be formed in $dom^c(D)$ as outlined in Lemma 2.2. This produces $P_i$ on the vertices $u_1, ..., u_i$ and $P_j$ on the vertices $v_1, ..., v_j$ in $dom^c(D)$.

($\Rightarrow$) Given a biorientation of the edges in $UG(P_i)$ and $UG(P_j)$ pursuant to Lemma 2.2, we know that $P_i$ and $P_j$ are formed on vertices $u_1, ..., u_i$ and $v_1, ..., v_j$, respectively in $dom^c(D)$. Since all other edges are bidirected, all other pairs of vertices dominate, and no additional edges are created in $dom^c(D)$. Thus, $dom^c(D) = P_i \cup P_j$ so that $UG(D) \cong dom(D)$, and $u_2v_1$ is not an edge for any $1 \leq k \leq i$, $1 \leq l \leq j$. ■

Now we turn to the more interesting characterization. That where $dom^c(D)$ contains at least one edge between $V(P_i)$ and $V(P_j)$. We know that only four vertices may be the origin of oriented edges in $D$, and each is the source of at most one edge in $dom^c(D)$. In order to construct $dom^c(D)$ so that it is isomorphic to $UG^c(D)$ where vertices from both $V(P_i)$ and $V(P_j)$ are used in each path, it is helpful to detail what edges are created in $dom^c(D)$ given an oriented edge.

Lemma 5.3 Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be components of $UG^c(D)$ where $i, j \geq 3$.

1. If $(u_1, v_k)$ for $1 \leq k \leq j$ is an oriented edge in $D$, then $u_2v_k$ is an edge in $dom^c(D)$.
2. If $(v_1, u_k)$ for $1 \leq k \leq j$ is an oriented edge in $D$, then $u_i-1v_k$ is an edge in $dom^c(D)$.
3. If $(u_k, v_j)$ for $1 \leq k \leq i$ is an oriented edge in $D$, then $u_kv_j$ is an edge in $dom^c(D)$.
4. If $(v_j, u_k)$ for $1 \leq k \leq j$ is an oriented edge in $D$, then $u_kv_{j-1}$ is an edge in $dom^c(D)$.

**Proof.** In each case, the edge created in $dom^c(D)$ is between the one vertex not adjacent to the origin of the oriented edge and the vertex dominated by that same vertex. They do not dominate, and thus form an edge in $dom^c(D)$. ■

A consequence of this lemma is that some edges between the paths $U_iV_j$ can be formed in $dom^c(D)$ in two ways, or one way, or cannot be formed. An example of edges that cannot exist was given in Lemma 4.2. The previous lemma only addresses edges formed between the two sets of vertices. Sources for edges within each set were given in Lemma 2.2. The following corollary restates the results in Lemma 5.3 in terms of the number of ways in which edge in $dom^c(D)$ can be formed using oriented edges in $D$. This is important information for proving the final characterizations.

**Corollary 5.4** Let $P_i = u_1, ..., u_i$ and $P_j = v_1, ..., v_j$ be components of $UG^c(D)$ where $i, j \geq 3$. If $(u_k, v_j)$ is an edge in $dom^c(D)$ where $k = 2i-1$ and $l = 2i-1$, then there are two possible sources for the edge. All other edges of the form $(u_k, v_j)$ in $dom^c(D)$ have at most one source.

The structure of $D$ supersedes the possible labelings of the vertices, so the focus is on the relationships of the oriented edges to each other. In this way, the isomorphic labelings are incorporated into the final results. Of course, to be able to indicate which vertices are involved with the oriented edges in $D$ and the edges in $dom^c(D)$, some labeling convention must be adopted. Thus, we will let $u$ and $u'$ be the end vertices of one path, with $v$ and $v'$ the end vertices of the other path. At times, we will need to discuss the vertices that are adjacent to $u$, $u'$, $v$, and/or $v'$ in $UG^c(D)$. Thus, if $u = u_1$ or $u_i$, then $u_kv_j$ or $u_{i-1}v_i$ respectively. Likewise, if $v = v_1$ or $v_j$, then $v_iu_k$ or $v_{j-1}u_i$ respectively. Similar labelings will be utilized for $u_i'$ and $v_j'$.

From Theorem 4.4, the cases we need to consider in our characterization of $D$ where $UG^c(D) = P_i \cup P_j$, are $1) j = i$, $2) j = i + 1$, $3) j = 2i - 1$, $4) j = 2i$, and $5) j = 2i + 1$. First, we consider the case where $i = j$ is odd. The only oriented edges will be between the end vertices of the original two paths. Therefore, there are two oriented edge formations possible. These are shown in Figure 5.
Theorem 5.5 Let $UG^c(D) = P_i \cup P_j$, where $i = j \geq 3$ are odd. Further, let $P_i = u, u_k, \ldots, u_k, u'$ and $P_j = v, v_i, \ldots, v'$. In $UG^c(D)$. $UG(D) \cong dom(D)$, where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$ if and only if every edge of $UG(D)$ is bidirected in $D$ except for the following.

1. $(u,v)$ is an oriented edge of $UG(D)$ in $D$, and
2. exactly one of $(v,u')$ or $(v',u)$ is an oriented edge of $UG(D)$ in $D$, and
3. for $i, j \geq 5$, oriented edges may be formed as stated in Corollary 2.3 such that $u, v, u'$, and $v'$ are each the origin of at most one oriented edge.

Proof. (\implies:) Pursuant to Lemma 2.1, there is one odd generated subpath and one even generated subpath on each of $V(P_i)$ and $V(P_j)$ in $dom^c(D)$. To create two paths of length $i$, each odd path must have an edge to an even path. Since edges must exist between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, the odd subpath generated on $V(P_i)$ must have an edge to the even subpath generated on $V(P_j)$, and similarly for their counterparts. Say that $uw_i$ is any such an edge in $dom^c(D)$, forming path $u, \ldots, u_k, u'$. Then $(u,v)$ must be an oriented edge in $D$. This results in $uw_i$ never being an edge in $dom^c(D)$ when $UG(D) \cong dom(D)$. To create the other path in $dom^c(D)$, we must form edge $v'w_i, v'v_i$, or $v'v_i$. Edges $v'w_i$ and $v'v_i$ are formed by oriented edges $(v',u)$ and $(v,u')$ in $D$ respectively. These correspond to isomorphic digraphs. Thus, we only need list $(v,u')$. This relationship is shown in the first digraph of Figure 5. Finally, if edge $u'v_i$ is in $dom^c(D)$, $(v',u')$ must be an oriented edge of $D$. Thus, within isomorphic labeling, $(u,v)$ must be an oriented edge along with one of $(v,u')$ or $(v',u)$ since $UG(D) \cong dom(D)$. Additionally, oriented edges listed in part (3) may be created as stated in Corollary 2.4 as long as no vertex is the origin of more than one oriented edge.

(\impliedby:) If $(u,v)$ and $(v,v')$ are oriented edges in $D$, then edges $ukv$ and $v'v_i$ are formed in $dom^c(D)$, creating two paths with $i$ vertices each. If $(u,v)$ and $(v',u')$ are oriented edges in $D$, then edges $ukv$ and $v'v_i$ are formed in $dom^c(D)$, creating two paths with $i$ vertices each. In both cases, $UG(D) \cong dom(D)$. Any directed edge in part (3) that does not create a vertex that is the origin of more than one oriented edge is allowed, with no additional edges formed. Therefore, $dom^c(D) = P_i \cup P_j$ where $i = j$ are odd, and $UG(D) \cong dom(D)$.

Next we examine the case where $i = j$ is even. To do so, it is easiest to separate possible orientations into classes determined by the structure of $dom^c(D)$. We will describe digraphs where the paths in $dom^c(D)$ have edges where 1) two end vertices are used in the two adjoining edges, 2) no end vertices are used in the two adjoining edges, and 3) one end vertex is used in the two edges.

In determining the digraphs that result in isomorphic underlying and domination graphs, we find that there is one formation that is not allowed in $D$ when we look at the first case listed above. That where two end vertices are in the two edges of $dom^c(D)$. The following lemma shows that in this case in $D$, the oriented edges will always be from one set of vertices, $V(P_i)$, to the other set, $V(P_j)$, where the paths are arbitrarily labeled.

Lemma 5.6 Let $UG^c(D) = P_i \cup P_j$ with $i = j$ even and $UG(D) \cong dom(D)$. Further, let $x$ and $w$ be end vertices in $UG^c(D)$. If $xy$ and $wz$ are edges between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, then $x$ and $w$ are both end vertices of $P_i$ or both end vertices of $P_j$.

Proof. Suppose that $x$ and $w$ are in separate vertex sets. Let $x = u$ in $P_i$ and $w = v$ be the other end vertex. Also, let $v = w$ in $P_j$ and $v'$ be the other end vertex. Then $u, \ldots, u_k, u', v, v_i, \ldots, v'$ and $v_i, \ldots, v'$ are the four generated subpaths in $dom^c(D)$. Since $xy = uy$ and $wz = vz$ are edges in $dom^c(D)$, $y \neq v'$ and $z \neq u'$, else paths $u', \ldots, v_i, v, \ldots, u_k$ and $v_i, \ldots, v'$ are formed, and $u$ and/or $v$ appears in more than one path in $dom^c(D)$. Thus, $y = u_i$ and $z = u_k$. But then there is only one way to create edge $uw_i$ in $dom^c(D)$, and that is with the oriented edge $(v,v)$ in $D$. Likewise, the one way to form edge $uv_i$ in $dom^c(D)$ is with the oriented edge $(u,v)$ in $D$. They cannot be used together, as they form a bidirected edge. Thus, both $x$ and $w$ must be end vertices of either $P_i$ or $P_j$.

With the preceding lemma, it is now possible to list the oriented edges that may occur in $D$ when $i = j$ is even. Figure 6 illustrates the possible oriented edges that may occur in $D$ given the number of end vertices in $UG^c(D)$ that are used to connect the paths in $dom^c(D)$.

Theorem 5.7 Let $UG^c(D) = P_i \cup P_j$, where $i, j \geq 3$ are even. Further, let $P_i = u, u_k, \ldots, u_k, u'$ and $P_j = v, v_i, \ldots, v'$. In $UG^c(D)$. $UG(D) \cong dom(D)$, where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $UG(D)$ is bidirected in $D$ except for the following.

1. $(u,v)$ and $(v,v')$ are both oriented edges in $D$, or
   i. $(u,v)$ or $(u,v_k)$ or both are oriented edges in $D$, and
   ii. $(v',v_i)$ or $(v',u_k)$ or both are oriented edges in $D$, or
2. $(u,v)$ is an oriented edge in $D$, and
   i. $(u,v)$ or $(u,v_k)$ or both are edges in $D$, and
   ii. $(v',v_i)$ or $(v',u_k)$ or both are edges in $D$, and
2. for i, j ≥ 4, oriented edges may be formed as stated in Corollary 2.3 such that u, u', v, and v' are each the origin of at most one oriented edge.

The proof of Theorem 5.7 can be found in the last section of this paper.

To continue the characterization, we observe the case where j = i + 1. The values for i and j alternate odd and even, which in practice does not make a difference with the results. However, it is important to understand when we are dealing with the even path and when the odd path is discussed. While we could split the results into odd and even, that is not necessary if we generalize the paths. So for this case, we will let $P_2 = e, e_k, \ldots, e_k, e'$ be the even path in $U G^e(D)$, and $P_0 = o, o_1, \ldots, o_1, o'$ be the odd path. Thus, one generated subpath in $dom^c(D)$ has end vertices e and $e'$. They can only be joined to an interior vertex, $o_1$ or $o'_1$, of the other path to create the odd path in $dom^c(D)$. There is only one distinct way to do that. This narrows down the choices. Once the choice is made, as represented by the directed edge from the upper left to the lower left corner of every digraph in Figure 7, there is a variety of ways to produce the even path in $dom^c(D)$.

![Digraphs with oriented edges](image)

Figure 7: Digraphs where $P_i$ and $P_j$ are formed in $dom^c(D)$ when $j = i + 1$ and there is an edge between $V(P_i)$ and $V(P_j)$. All are shown with the maximum number of oriented edges between $V(P_i)$ and $V(P_j)$. The even set of vertices is on the top, and bidirected edges are omitted.

Theorem 5.8. Let $U G^c(D) = P_i \cup P_j$, where $i \geq 3$ and $j = i + 1$. Further, let $P_2 = e, e_k, \ldots, e_k, e'$ be the even path in $U G^c(D)$, and $P_0 = o, o_1, \ldots, o_1, o'$ be the odd path. $U G(D) \cong dom(D)$ where there is an edge between $V(P_i)$ and $V(P_j)$ in $dom^c(D)$, if and only if every edge of $U G(D)$ is bidirected in D except for the following.

1. $(e, o)$ is an oriented edge in $D$, and
   (a) $(e', e)$ is an oriented edge in $D$, or
   (b) $(e_1, o)$ or $(e', e_k)$ or both are oriented edges in $D$, or
   (c) $(e_1, o_1)$ or $(e, e_k)$ or both are oriented edges in $D$, and

2. oriented edges may be formed as stated in Corollary 2.3 such that $e, e', o$, and $o'$ are each the origin of at most one oriented edge.

Proof. Part (2) is valid by Corollary 2.4. The remaining arguments deal with part (1).

(\(\implies\)) If we let o be either of the end vertices of the odd path in $U G^c(D)$, then let $e_1 o$ be the edge formed in $dom^c(D)$, we obtain an arbitrary labeling similar to that in the proof of Theorem 5.7. Path $P_0 = o, o_1, \ldots, e_k, e'$ is the odd path formed in $dom^c(D)$. The arguments here follow the same logic as the proofs in Theorems 5.5 and 5.7. However, once we have chosen o and $e_k$, there is a selection for how to form the remaining path in $dom^c(D)$. Figure 7(a) illustrates the option of having the one end vertex, e, that is not in $P_1$ as one of the vertices incident with the edge that connects the remaining two subpaths. The only way for this to occur is for $(e', e)$ to be an oriented edge in $D$. The remaining two possible connecting edges, $e_k o_1$ and $e' o_1'$, can be formed in two ways each, as listed in parts 1(b) and 1(c) of the theorem statement. These two options are not isomorphic, as the relationships of the oriented edges are different, as seen in Figure 7(b) and (c).

(\(\implies\)) By Lemma 5.3, if $(e, o)$ is an arc, then edge $e_1 o$ is in $dom^c(D)$, and each of parts (a) through (c) creates an edge connecting the remaining two subpaths in $dom^c(D)$. This result in $dom^c(D)$ consisting of two disjoint paths with i and j = i + 1 vertices. Thus, $U G^c(D) \cong dom^c(D)$ and $U G(D) \cong dom(D)$.

All of the previous cases have dealt with paths $P_i$ and $P_j$ in $dom^c(D)$ that were created by connecting two subpaths for each. Now we turn our attention to those cases where $P_i$ is one of the generated subpaths in $dom^c(D)$, and $P_j$ is created by connecting the remaining three subpaths.

Remark 5.9. If $j = 2i - 1$, 2i, or 2i + 1, where $U G(D) \cong dom(D)$, then $P_i$ in $dom^c(D)$ is a generated subpath on $V(P_j)$.

We begin now with the case where $j = 2i - 1$. Figure 8 shows examples of part (1) in the following theorem. The digraphs are shown on the same set of vertices as their associated $dom^c(D)$ graphs. Edges shown are those for $dom^c(D)$, and bidirected edges of $D$ are omitted for simplicity. Labeling on part (a) shows the labeling convention adopted.
Figure 8: Examples of digraphs and their associated \( \text{dom}^e(D) \) graphs where \( i \) is odd, \( j = 2i - 1 \), and \( P_i \) is a generated subpath on \( V(P_j) \). Edges shown are in \( \text{dom}^e(D) \), while arcs are in \( D \). Bidirected edges are omitted.

Theorem 5.10 Let \( UG^e(D) = P_1 \cup P_j \), where \( i \geq 3 \) and \( j = 2i - 1 \). Further, let \( P_i = u, u_k, ..., u_k', u' \) and \( P_j = v, v_1, ..., v_1', v' \) in \( UG^e(D) \). \( UG(D) \cong \text{dom}(D) \) where there is an edge between \( V(P_i) \) and \( V(P_j) \) in \( \text{dom}^e(D) \), if and only if every edge of \( UG(D) \) is bidirected in \( D \) except for the following.

1. If edges are oriented in \( V(P_i^e) \) as stated in Lemma 2.2, then
   (a) \((v, u)\) is an oriented edge in \( D \), or
   (b) if \( i \) is odd, then \((u, u_k)\) or \((v, u'_k)\) or both are oriented edges in \( D \).

2. If edges are not oriented in \( V(P_i^e) \) as stated in Lemma 2.2, then
   (a) \((v, u)\) is an oriented edge in \( D \), and
      i. \((u, u')\) or \((v, u_k)\) or both are oriented edges in \( D \), or
      ii. if \( i \) is odd, then \((u', u_k')\) or \((v', u'_k)\) or both are oriented edges in \( D \), or
      iii. if \( i \) is even, then \((v', u')\) is an oriented edge in \( D \), or
   (b) if \( i \) is even, then
      i. \((u, u_k)\) or \((v, u')\) or both are oriented edges in \( D \), and
      ii. \((u', u_k')\) or \((v', u'_k)\) or both are oriented edges in \( D \).

3. If \( i \) is odd or even, oriented edges stated in Corollary 2.3 may be used in addition to the required arcs in (1) and (2) as long as \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge in \( D \).

The proof of Theorem 5.10 can be found in the final section of this paper.

Continuing with the cases where \( P_i \) in \( \text{dom}^e(D) \) is a generated subpath, we proceed to \( j = 2i \). Unlike the cases where \( j = 2i + 1 \) or \( 2i - 1 \), here \( P_j \) is on an even number of vertices. This allows the choice of which generated subpath, \( V_1 \) or \( V_2 \), will be \( P_i \) in \( \text{dom}^e(D) \). Generally, we will let \( P_i = V_1 = v_1, ..., v' \) and \( P_j = U_1, v_1, ..., v, v_2 \), where \( U_1 = u, u'_k \) and \( U_2 = u_k, u' \).

Figure 9: A digraph \( D \) where \( j = 2i \) on the same set of vertices as \( \text{dom}^e(D) \). Edges shown are those in \( \text{dom}^e(D) \), while arcs are in \( D \). Bidirected edges of \( D \) are omitted.

Theorem 5.11 Let \( UG^e(D) = P_1 \cup P_j \), where \( i \geq 3 \) and \( j = 2i \). Further, let \( P_i = u, u_k, ..., u_k', u' \) and \( P_j = v, v_1, ..., v_1', v' \) in \( UG^e(D) \). \( UG(D) \cong \text{dom}(D) \) where there is an edge between \( V(P_i) \) and \( V(P_j) \) in \( \text{dom}^e(D) \), if and only if every edge of \( UG(D) \) is bidirected in \( D \) except for the following.

1. If edges are oriented in \( V(P_i^e) \) as stated in Lemma 2.2, then
   (a) \((v', u)\) is an oriented edge in \( D \), or
   (b) if \( i \) is odd, then \((u, v')\) is an oriented edge in \( D \).

2. If edges are not oriented in \( V(P_i^e) \) as stated in Lemma 2.2, then
   (a) if \( i \) is odd, then \((v', u)\) is an oriented edge in \( D \), and
      i. \((u, v)\) is an oriented edge in \( D \), or
      ii. \((u', v)\) is an oriented edge in \( D \), or
   (b) if \( i \) is even, then \((u, v)\) is an oriented edge in \( D \) and
      i. \((v', u)\) is an oriented edge in \( D \), or
      ii. \((u', v)\) or \((v', u'_k)\) or both are oriented edges in \( D \).

3. If \( i \) is odd or even, oriented edges stated in Corollary 2.3 may be used in addition to the required arcs in (1) and (2) as long as \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge in \( D \).
6 Proofs of Selected Results Omitted Earlier

Proof. (\(\implies\)) The case where \(j = 1\) is obvious as the two vertices always dominate in \(D\).

When \(j = 2\), \(E(UG(D)) = \{u_{21}, u_{22}\}\). If \((u_{21}, u)\) is an orientation and \((u_{22}, v_{i})\) is not, then \(u_{21}\) and \(v_{22}\) do not dominate, and form \(P_{3}\) in dom\(^{e}\)(\(D\)) with \(v_{1} = P_{1}\). Likewise, if \((u_{22}, u)\) is an orientation and \((v_{1}, v)\) is not, then \(u_{22}\) and \(v_{1}\) form \(P_{3}\) in dom\(^{e}\)(\(D\)) with \(v_{2} = P_{2}\). If \((u, v_{1})\) and \((u, v_{2})\) are oriented edges, then \(v_{1}v_{2}\) forms \(P_{3}\) in dom\(^{e}\)(\(D\)) with \(u = P_{1}\). Thus, \(UG(D) \cong dom(D)\).

When \(j \geq 3\), for part (a), Lemma 2.2 gives us constructions that result in the formation of \(P_{j}\) in dom\(^{e}\)(\(D\)). Since \(u\) dominates all of the \(v_{i}\), no relationships between the \(v_{i}\) are changed, and \(u_{21}\) is an edge in dom\(^{e}\)(\(D\)). Thus, \(u = P_{1}\) and the \(u_{2j}\) form \(P_{j}\) in dom\(^{e}\)(\(D\)), giving \(UG(D) \cong dom(D)\) and \(UG(D) \cong dom(D)\).

For part (b), if \((u, v_{p})\) and \((u, v_{q})\) are the only orientations of the edges of \(UG(D)\), then the only edges formed in dom\(^{e}\)(\(D\)) without the source \(u\) are the generated subpaths. The vertex \(u\) is the source for edge \(v_{p}v_{q}\) in dom\(^{e}\)(\(D\)). There is no vertex in \(P_{j}\) that is not adjacent to \(u\), so only the edge \(v_{p}v_{q}\) is formed. Since \(v_{p}v_{q}\) joins the two subpaths in dom\(^{e}\)(\(D\)), forming \(P_{j}\), and \(u\) is \(P_{1}\) for reasons explained in part (a), \(UG(D) \cong dom(D)\).

For part (c), let the edges between vertices \(v_{i}\) be oriented as in Lemma 2.2. First, consider the additional orientation \((u, v_{k})\) for \(k = 1, \ldots, j\). The vertex \(u\) is the source for any edge in dom\(^{e}\)(\(D\)) since every vertex other than \(v_{k}\) dominate it. Thus, only the edges in dom\(^{e}\)(\(D\)) formed by oriented edges specified in Lemmas 2.2 and 2.1 are created, and \(u\) is an isolated vertex. Therefore, dom\(^{e}\)(\(D\)) = \(P_{j} \cup P_{2}\) and \(UG(D) \cong dom(D)\). Now consider that \(u\) is the origin of two oriented edges \((u, v_{k})\) and \((u, v_{j})\) where \(k = 1, \ldots, j - 2\) and \(l = k + 2\). Then \(u\) is a source of the edge \(v_{k}v_{l}\) in dom\(^{e}\)(\(D\)), for which \(v_{k+1}\) is also a source since it is not adjacent to either vertex. Or, if \(k = p\) and \(l = q\), then \(u\) is a source for the edge \(v_{k}v_{q}\) in dom\(^{e}\)(\(D\)), for which a vertex \(v_{l}\) is also a source, as determined in Lemma 2.2. In either case, \(u\) is a source formed in dom\(^{e}\)(\(D\)), \(u\) is an isolated vertex in dom\(^{e}\)(\(D\)), and the vertices \(v_{k}\) form the path \(P_{j}\). Thus, \(UG(D) \cong dom(D)\).

(\(\implies\)) The case where \(j = 1\) is obvious as \(UG(D) = K_{2}\) and the two vertices always dominate in \(D\).

When \(j = 2\) and \(UG(D) \cong dom(D)\), then there must be a source to one edge in dom\(^{e}\)(\(D\)). A vertex cannot be the source of any edge with which it is incident. Therefore, if it is possible, \(u\) must be the source for \(v_{1}v_{2}\), \(v_{1}\) must be the source for \(v_{2}v_{3}\) and \(v_{2}\) must be the source for \(u_{1}\). For the first case, \(u\) must dominate both \(v_{1}\) and \(v_{2}\) in order to be the source. In the second case, \(v_{1}\) and \(v_{2}\) are not adjacent, so the orientation \((v_{1}, u)\) of \(u_{1}\) makes \(uv_{1}\) an edge in dom\(^{e}\)(\(D\)) since neither \(u\) nor \(v_{2}\) dominates \(v_{1}\). However, \((u, v_{2})\) must be an oriented edge in \(D\), so that \(uv_{1}\) is an edge in \(UG(D)\). A similar argument holds for the case where \(v_{2}\) is the source for \(uv_{1}\).

When \(j \geq 3\), Lemma 2.2 shows a construction that creates an isomorphic
copy of $P_j$ in $\text{dom}^c(D)$ for $j \geq 3$. Lemma 3.1 guarantees that $u$ must be equal to $P_1$ in $\text{dom}^c(D)$ since $UG(D) \cong \text{dom}(D)$. Therefore, no $v_j$ will be the origin of an oriented edge with terminal vertex $u$, since edge $w_{i+1}$ or $w_{i-1}$ would be created in $\text{dom}^c(D)$. Thus, $u$ will be the only possible origin for any additional oriented edges outside of those created by Lemma 2.2. The vertex $u$ will either be a source vertex, or it will not be. If it is not, since $u$ is adjacent to all vertices in $UG(D)$, either all edges $w_i$ for $i = 1, \ldots, j$ will be bidirected, proving part 3(a) above, or exactly one will have the orientation $(u, v_j)$. If not, $u$ would be a source of an edge in $\text{dom}^c(D)$. This proves part 3(c)(i) above.

If $u$ is a source for an edge in $\text{dom}(D)$ and we do not use Lemma 2.2 to create path $P_j$ in $\text{dom}(D)$, then $u$ must be the source for edge $v_j v_{j+1}$ so that $P_j$ is created in $\text{dom}(D)$. Since $u$ is adjacent to both vertices in $UG(D)$, both $(u, v_j)$ and $(u, v_{j+1})$ must be oriented edges in $D$, proving part 2.

If $u$ is a source for an edge in $\text{dom}(D)$ and we use Lemma 2.2 to create $P_j$ in $\text{dom}(D)$, then $u$ must be the source for an edge that is in $P_j$. Thus, $u$ must be the source of an edge $v_i v_{i+2}$, for $i = 1, \ldots, j-2$, or $v$ must be the source of edge $v_j v_{j+1}$. Since $u$ is adjacent to all vertices in $UG(D)$, $(u, v_j)$ and $(u, v_{j+1})$ or $(u, v_j)$ and $(u, v_{j+1})$ must be oriented edges in $D$, proving part 3(c)(ii).

Lemma 2.6 guarantees that $u$ cannot be the source of more than one edge in $\text{dom}^c(D)$, so it is the origin of at most two oriented edges in $D$.

Following is the proof for Theorem 3.7.

Proof. (\(\Rightarrow\)) By Lemma 2.10, when $UG(D) \cong \text{dom}(D)$, $u u'$ is not an edge in $\text{dom}^c(D)$. Therefore, either $v_i v_j$, $v_j v_{j+1}$, or $v_{j+1} v_{j+2}$ must be $P_j$ in $\text{dom}(D)$. Since $u$ can be either vertex $u_2$ or $u_3$, the second two choices reduce to $u v_{j+1}$. Edge $v_i v_{j+1}$ is a generated subpath in $\text{dom}^c(D)$, so if $v_i v_{j+1} = P_j$ in $\text{dom}(D)$, edges $u v_{j+1}$ and $v_{j+2} v_{j+1}$ must also be in $\text{dom}^c(D)$ forming $P_j$. From Lemma 2.8 we know that only vertices $u$, $v$, $v'$, and $v''$ may be the sources of additional edges in $\text{dom}^c(D)$. Edge $u v_{j+1}$ can be generated with oriented edges $(u', v_j)$, $(v, u)$, or $(v', u)$. Likewise, edge $v_{j+1} v_{j+2}$ can be generated with oriented edges $(u, v_{j+1})$, $(v', u)$, or $(v', u')$. We may use any combination of the oriented edges to create each of the edges in $\text{dom}^c(D)$. At least one from each group must be in $D$ so that the associated edge is created in $\text{dom}(D)$. However, Lemma 2.9 restricts the number of edges we may orient. Therefore, only digraphs where $u$, $u'$, $v$, and $v'$ are the origin of at most one oriented edge of the preceding form are possible when $UG(D) \cong \text{dom}(D)$. Additionally, according to Lemma 3.6, edge $v_i v_{j+1}$ must be bidirected in $D$. There are no other sources for the given edges in $\text{dom}^c(D)$, so all other edges in $UG(D)$ must be bidirected in $D$.

If $u v_{j+1} = P_j$ in $\text{dom}(D)$, then $u v_j$ or $u' v_j$ must also be edges in $\text{dom}^c(D)$. Within isomorphic labeling, we will generate edge $u' v_j$. This may be done only by using oriented edge $(u, v')$. Note that $(v, u')$ will not produce the desired edge in this case, and there are no other vertices that may serve as the origin of an oriented edge since $u'$ and $v'$ cannot be the source of their own edge. To generate the edge $u v_{j+1}$, possible oriented edges are $(u', v_j)$, $(v', u)$ and $(v, u')$. However, since $(u, v')$ must be an oriented edge in every biorientation of $UG(D)$, and $UG(D) \cong \text{dom}(D)$, $(v', u)$ can never be used. Thus, we must have $(u, v')$ and any selection of the other two directed edges.

(\(\Rightarrow\)) The edge $v_i v_j$ is generated as stated in Lemma 2.1. Vertices $u'$, $v$ and/or $v'$ are sources for the edge $u v_{j+1}$ in $\text{dom}^c(D)$ when $D$ has oriented edges $(u', v_j)$, $(v, u)$ and/or $(v', u)$ respectively. No other edges in $\text{dom}^c(D)$ are generated by these directed edges. Likewise, vertices $u'$, $v$ and/or $v'$ are sources for the edge $v' v_j$ in $\text{dom}^c(D)$ when $D$ has oriented edges $(u, v_j)$, $(u', v')$ and/or $(v', u')$ respectively. As long as each of $u$, $u'$, $v$, and $v'$ are the origin of at most one oriented edge in $D$, any combination of these oriented edges with at least one from each group results in $\text{dom}^c(D) = w_i u v_j v_{j+1}$, and $UG(D) \cong \text{dom}(D)$.

If $(u, v)$ is an oriented edge in $D$, then $u v'$ is an edge in $\text{dom}(D)$, and $(v', u)$ is a path on three vertices. When $D$ has oriented edges $(u', v)$ and/or $(v', u')$, edge $u v'$ is formed in $\text{dom}^c(D)$. Any digraph $D$ with one or both of these oriented edges of $UG(D)$ will have the edge $u v'$. Thus, $\text{dom}^c(D) = u v_j v_{j+1}$, $u v_{j+1} v_{j+2}$, and $UG(D) \cong \text{dom}(D)$.

Following is the proof for Theorem 3.8.

Proof. (\(\Rightarrow\)) Paths $v_i v_j$ and $v_j v_{j+1}$ are generated subpaths in $\text{dom}^c(D)$. Since $u v_i$ cannot be an edge in $\text{dom}^c(D)$, either $v_i v_j$ or $v_j v_{j+1}$ form $P_j$ when $UG(D) \cong \text{dom}(D)$. Since the choice of $v$ is arbitrary, say that $v_i v_j = P_j$ in $\text{dom}^c(D)$, then $u v_i$, $v_i v_{j+1}$, or $v_{j+1} v_j = P_j$ in $\text{dom}^c(D)$. Thus, edges $u v_i$ and $u v_{j+1}$ must be formed. The vertex $v'$ is an end vertex. The only vertex not adjacent to $v'$ in $UG(D)$ cannot be used as the source of an edge in $\text{dom}^c(D)$. Therefore, the only directed edge that can be used to form $u v_i$ is $(u', v')$. So $(u', v')$ must be an oriented edge in every biorientation of $UG(D)$, proving part 1.

To form edge $u v_i$ in $\text{dom}(D)$, we are not restricted in the same way as that for edge $u v_i$. Here, $v_i$ is an interior vertex. Vertex $v$ is not adjacent to $v_i$ in $UG(D)$, and may be the origin of an oriented edge in $D$. Therefore, the oriented edges that can be used separately or together to form edge $u v_i$ in $\text{dom}^c(D)$ are $(u, v_i)$ and $(u', v_i)$, proving part 2.

Part (3) follows from Corollaries 2.3 and 2.4 as well as Lemma 2.9.

(\(\Rightarrow\)) Paths $v_i v_{j+1}$ and $v_{j+1} v_j$ are generated subpaths in $\text{dom}^c(D)$. Vertex $u v_i$ is the source for edge $u v_{j+1}$ in $\text{dom}(D)$ when $(u', v_i)$ is an oriented edge in $D$. If $(u, v_i)$ and/or $(v_i, v_j)$ are oriented edges in $D$, edge $u v_{j+1}$ is created in $\text{dom}^c(D)$. Vertex $v_i$ is a source of edge $v' v_j$ in $\text{dom}^c(D)$ when $(v_i, v')$ is an oriented edge in $D$, and $v' v_j$ also has the source $v_j$, which is neither vertex is adjacent to $v_i$ in $UG(D)$. Likewise, $v'$ is a source of edge $v' v_j$ when $(v', v)$ is an oriented edge in $D$, where $v_i$ is always a source for $v' v_j$. Thus, if either or both of these edges is in $D$, no new edges appear in $\text{dom}^c(D)$. Since all other edges of $UG(D)$ are bidirected in $D$, there are no other edges formed in $\text{dom}^c(D)$. Thus, $\text{dom}^c(D) = v_i u v_j v_{j+1}$, and $UG(D) \cong \text{dom}(D)$.

Following is the proof for Theorem 4.4.

Proof. (\(\Rightarrow\)) If $i$ or $j$ is even, then there are two generated subpaths, each of length $\frac{i}{2}$ or $\frac{j}{2}$ in $\text{dom}^c(D)$. If $i$ or $j$ is odd, then the two generated subpaths in $\text{dom}^c(D)$ are of length $\frac{i+1}{2}$ and $\frac{j+1}{2}$, or $\frac{i+1}{2}$ and $\frac{j+1}{2}$ respectively. When $UG(D) \cong \text{dom}(D)$ and we want edge $u v_i$ to be in $\text{dom}^c(D)$, we must be
able to connect each path formed with vertices \( u_1, \ldots, u_i \) to a path formed with vertices \( v_1, \ldots, v_j \).

First, we concentrate on each subpath on \( V(P_i) \) being connected to a different subpath on \( V(P_j) \). Consider \( i \) and \( j \) both even. Then we must have \( \frac{1}{2} + \frac{1}{2} = i \) or \( j \), so \( j = i \).

Consider \( i \) and \( j \) both odd. From Corollary 4.3, we know that we cannot connect the two odd paths in \( dom^c(D) \). Therefore, the odd subpath generated on vertices \( u_1, \ldots, u_i \) must augment the shorter even subpath generated on vertices \( v_1, \ldots, v_j \). The same holds true for the other odd subpath. Thus, the only time that \( \frac{1}{2} + \frac{1}{2} = i \) and \( \frac{1}{2} + \frac{1}{2} = j \) (or vice versa) when \( i = j \).

Consider one of \( i \) or \( j \) odd. Say \( i \) is odd. Since \( u_4v_3 \) is in \( dom^c(D) \), \( u_1, u_2, u_3, u_4 \) must be connected to a path that is the same length as the path that \( v_1, v_2, v_3, v_4 \) is from the same set of vertices. We know that \( \frac{1}{2} + \frac{1}{2} = i \) and \( \frac{1}{2} + \frac{1}{2} = j \). Thus, \( j = i + 1 \).

Now we concentrate on the possibility that one subpath is not connected to another subpath. Since \( i \leq j \), this subpath must form \( P_i \) in \( dom^c(D) \). Thus, it is one of the generated subpaths formed on \( V(P_i) \). The length of subpath on \( V(P_i) \) is \( \frac{1}{2} \) if \( j \) is even, or \( \frac{1}{2} + \frac{1}{2} = j \) if \( j \) is odd. Thus, \( j = 2i, 2i + 1, \) or \( 2i - 1 \), respectively.

(\( \Rightarrow \)) Constructions that do not depend upon this theorem are given in Theorems 5.5, 5.7, 5.8, 5.10, 5.11, and 5.12, which take the values for \( i \) and \( j \) given in the statement of Theorem 4.4 and give biorientations of the underlying graph resulting in \( UG(D) \approx dom(D) \) with an edge between \( V(P_i) \) and \( V(P_j) \).

Thus, there are biorientations of the edges of \( UG(D) \) such that \( UG(D) \approx dom(D) \) where there are edges between the \( u_i \) and the \( v_j \).

Following is the proof for Theorem 5.7.

Proof. Part (2) is valid by Corollary 2.4. The remaining arguments deal with part (1).

(\( \Rightarrow \)) The four generated subpaths are of the same length in \( dom^c(D) \). Since there is an edge between \( V(P_i) \) and \( V(P_j) \) in \( dom^c(D) \), each of these paths on a subset of \( V(P_i) \) must be connected to a path from \( V(P_j) \) to form the isomorphic graph. We know that \( w, u, u', v, v' \) cannot be edges in \( dom^c(D) \). Thus, only two, zero or one of \( u, u', v, v' \) can be edges with an edge connecting the two sets of vertices in \( dom^c(D) \). We will separate the characteristic into these three possibilities.

If two end vertices are used, let \( v \) be one of them. From Lemma 5.6, \( v' \) must be the other of the two vertices since they must be from the same set of vertices, \( V(P_i) \) or \( V(P_j) \). Any edge containing \( v \) or \( v' \) in \( dom^c(D) \) will have \( u \) or \( u' \) as a source. Let \( u \) be either vertex, and \( u \) or \( v \) an oriented edge in \( D \). Then \( (u', u'') \) must be the other oriented edge so that two edges incident to \( v \) and \( u' \) are created in \( dom^c(D) \). The arbitrary selection of \( u \) and \( v \) includes all possible labelings that produce this. Thus, edges \( u_1u_2 \) and \( u'u_2 \) are in \( dom^c(D) \), creating two paths that are disjoint, with each oriented edge. Figure 6(a) shows the generic orientation of the only two edges that can accomplish this.

If there are no end vertices connecting \( P_i \) and \( P_j \) in \( dom^c(D) \), then all of the vertices \( u_k, u'_k, v_l, \) and \( v'_l \) must be incident with the two edges between vertex sets. Arbitrarily pick \( u_k \) to label one of the four possible interior vertices. Arbitrarily label one of the other interior vertices of the other vertex set as \( v_l \). So, in \( UG^c(D) \), we have paths \( u, u_k, \ldots, u'_k, u' \) and \( v, v_l, \ldots, v'_l, v' \). Say that \( u_kv_l \) and \( u'_kv'_l \) are the two edges in \( dom^c(D) \). This selection includes all possible edges between the two sets where no end vertices are used. Two paths with \( i \) vertices each are created. To form these edges in \( dom^c(D) \), Corollary 5.4 indicates that two oriented edges may be used, and we can use one or both of them in \( D \). Oriented edges \( (u, v_l) \) or \( (u, u_k) \) or both may be used to create edge \( u_kv_l \), while oriented edges \( (u', v') \) or \( (u', u'_k) \) or both may be used to create edge \( u'_kv'_l \). The possibility where all of these arcs are used is shown in Figure 6(b).

If there is one end vertex that is incident with an edge connecting \( V(P_i) \) and \( V(P_j) \) in \( dom^c(D) \), let us call that vertex \( v \). Choose \( u \) to be either end vertex of the other vertex set. Oriented edges \( (u, v) \) or \( (u', v) \) are the only ones that will produce an edge in \( dom^c(D) \) that is incident with \( v \). The arbitrary nature of the labeling of \( u \), allows us to reduce this to \( (u', v) \), producing edge \( u_kv_l \) in \( dom^c(D) \) and path \( u', u_k, v, u' \). So, \( (u, v) \) must be an oriented edge in \( D \). There are only two remaining interior vertices that are not on the path created by edge \( u_kv_l \) in \( dom^c(D) \). They are the vertices \( u'_k \) and \( v'_l \). They must form the second edge, creating the path of \( i \) vertices. This can be done if \( D \) contains oriented edges \( (u', v_l) \) or \( (u'_k, v) \) or both. Figure 6(c) gives the example where \( (u, u_k) \) and both of the other two oriented edges are in \( D \).

(\( \leftarrow \)) By Lemma 5.3, parts 1(a), (b) and (c), when used separately, create edges in \( dom^c(D) \) between \( V(P_i) \) and \( V(P_j) \) that result in \( dom^c(D) \) consisting of two disjoint paths with \( i = j \) even vertices each. Thus, \( UG^c(D) \approx dom^c(D) \) and \( UG(D) \approx dom(D) \)
the edge \( uv_1 \) can be used to connect \( U_1 \) to \( V_2 \), so \((v, u)\) can be an oriented edge in \( D \). When it is, \( U_2 \) must be connected to \( v'_2 \). When \( i \) is either odd or even, the edge \( uv'_1 \) does this, so \((u, v'_1)\) or \((v'_1, u)\) or both may be oriented edges in \( D \). Additionally, if \( i \) is odd, using \( uv'_1 \) instead of \( uv'_1 \) is a nonisomorphic construction connecting \( U_2 \) and \( V_2 \), where \((u', v'_1)\) or \((v'_1, u')\) or both are oriented edges in \( D \). Likewise, if \( i \) is even, using \( u'v'_1 \) instead of \( uv'_1 \) connects \( U_2 \) and \( V_2 \), where \((v', u')\) is an oriented edge of \( D \). This gives us part (2)(a) of the theorem.

All other choices when \( i \) is odd are isomorphic within the labeling. However, \( uv_1 \) does not have to be an edge if \( i \) is even. Both interior vertices, \( u_k \) and \( u'_k \), may be used to connect \( U_1 \) and \( U_2 \) respectively. Thus, edges \( u_kv_1 \) and \( u'_kv'_1 \) in \( dom^c(D) \) will create path \( P_j \). To do this, \( D \) must have oriented edges \((u, v_1)\) or \((v, u_k)\) or both, and oriented edges \((u', v'_1)\) or \((v', u'_k)\) or both. This gives us part (2)(b) of the theorem.

3. From Corollary 2.4, we are guaranteed that the arcs in part 3 will not alter the relationship \( UG(D) \equiv dom(D) \) as long as vertices \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge in \( D \).

\[ \Rightarrow \] In all constructions for parts (1) and (2), \( P_i \) is \( V_1 \) in \( dom^c(D) \). \( P_i \) is formed as follows, creating \( UG(D) \equiv dom(D) \) with an edge between \( V(P_i) \) and \( V(\tilde{P}_i) \). In part (1)(a), \( P_j = v_1, ... , v_n, U_1, U_2 \). In part (1)(b), \( P_j = U_1, U_2 \). In part (2)(a)(i), \( P_j = u', ... , u_k, V_2, U_2 \). In part (2)(a)(ii), \( P_j = u', ... , u, v_2, v_1, ... , v_k \). In part (2)(a)(iii), \( P_j = u_k, v_1, ... , v_n, U_1 \). In part (2)(b)(i), \( P_j = U_1, v_1, ... , v_n, U_2 \). In all cases, the oriented edges in Corollary 2.3 may be used and only generate edges already in the generated subpaths. Since \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge, \( UG(D) \equiv dom^c(D) \).

Following is the proof for Theorem 5.11.

**Proof.** The arbitrary selection of \( u \) and \( v \) generates all nonisomorphic digraphs.

\[ \Rightarrow \] \( UG(D) \equiv dom(D) \) so \( P_i \) in \( dom^c(D) \) must be the generated subpath \( V_1 = v_1, ... , v'_1 \) or \( V_2 = u_1, ... , u'_1 \). Without loss of generality, say \( P_i = V_2 \). Thus, subpaths \( V_1 \) and \( V_2 \) must form \( P_i \) in \( dom^c(D) \). Either \( U_1 \) and \( U_2 \) are connected by an edge or they are not.

1. Say that \( U_1 \) and \( U_2 \) are connected. This implies that oriented edges must be used as stated in Lemma 2.2. When \( i \) is odd, path \( U_e = u_1, ... , u'_k, u_k, ... , u' \) is formed. When \( i \) is even, path \( U_e = u_1, ... , u'_k, u_k, ... , u' \) is formed. To connect \( V_1 \) to \( U_e \) or \( U_e \) edge \( uv_1 \) can be formed when arbitrarily choosing \( u \) and \( v \). In both instances, oriented edge \((v, u)\) in \( D \) will create \( uv_1 \) in \( dom^c(D) \). There is no other way to connect \( V_1 \) to \( U_e \) since the only other vertex choices are end vertices that cannot form edges in \( dom^c(D) \). However, \( V_1 \) can be connected to \( U_e \) using edge \( u'v_1 \). Oriented edge \((u', v_1)\) in \( D \) is the only way to create that edge, and gives us part (1) of the theorem.

2. If \( U_1 \) and \( U_2 \) are not connected, this implies that \( V_1 \) must have one end vertex adjacent to \( U_1 \) and the other to \( U_2 \). When \( i \) is odd, this can be done nonisomorphically in two ways. In each, we create edge \( u_1v_1 \) by using oriented edge \((v', u)\) in \( D \). In addition, either \( u_k \) or \( v_k \) can be connected to \( v \). These are nonisomorphic choices since \( u_k \) is adjacent to our chosen \( u \) in \( UG^c(D) \), and \( v_k \) is not. We obtain edge \( uv \) or edge \( u'v' \) only by creating oriented edges \((u, v)\) or \((u', v')\) respectively in \( D \). This gives us part (2)(a) of the theorem. When \( i \) is even, at least one of \( u_k \) or \( v_k \) must be connected to \( v \) since neither \( u \) nor \( v' \) can be. Without loss of generality, say that \( u_kv \) is the edge, which implies the \((u, v)\) is an oriented edge in \( D \). The remaining subpath may connect to \( v_1 \) by either vertex \( u \) or \( u'_k \). Edge \( u_1v_1 \) is formed in one way, and that is by creating oriented edge \((v', u)\) in \( D \). Edge \( u'_k \) can be formed in two ways, using oriented edge \((u', v')\) or \((v', u'_k)\) or both. This gives us part (2)(b) of the theorem.

3. From Corollary 2.4, we are guaranteed that the arcs in part 3 will not alter the relationship \( UG(D) \equiv dom(D) \) as long as vertices \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge in \( D \).

\[ \Leftarrow \] In all constructions for parts (1) and (2), \( P_i = V_2 \) is an arbitrary choice. \( P_i \) is formed as follows, creating \( UG(D) \equiv dom(D) \) with an edge between \( V(P_i) \) and \( V(\tilde{P}_i) \). In part (1)(a), \( P_j = V_1U_1U_2 \). In part (1)(b), \( P_j = U_1U_2V_1 \). In part (2)(a)(i), \( P_j = u_1, ... , u_k, V_2, U_2 \). In part (2)(a)(ii), \( P_j = u', ... , u, v_2, v_1, ... , v_k \). In part (2)(a)(iii), \( P_j = u_k, v_1, ... , v_n, U_1 \). In part (2)(b)(i), \( P_j = U_1, v_1, ... , v_n, U_2 \). In part (2)(b)(ii), \( P_j = U_1, v_1, ... , v_n, U_2 \). In all cases, the oriented edges in Corollary 2.3 may be used and only generate edges already in the generated subpaths. Since \( u, u', v, \) and \( v' \) are each the origin of at most one oriented edge, \( UG(D) \equiv dom^c(D) \).

**References**


