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A Characterization of Connected $(1,2)$-Domination Graphs of Tournaments

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Abstract

Recently, Hedetniemi et al. introduced $(1,2)$-domination in graphs, and the authors extended that concept to $(1,2)$-domination graphs of digraphs. Given vertices x and y in a digraph D, x and y form a $(1,2)$-*dominating pair* if and only if for every other vertex z in D, z is one step away from x or y and at most two steps away from the other. The $(1,2)$-*dominating graph of* D, $dom_{1,2}(D)$, is defined to be the graph $G = (V, E)$, where $V(G) = V(D)$, and xy is an edge of G whenever x and y form a $(1,2)$-dominating pair in D. In this paper, we characterize all connected graphs that can be $(1,2)$-dominating graphs of tournaments.

Keywords: tournament, (1,2)-dominating pair, (1,2)-domination graph,

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The topic of domination, both in graphs and in digraphs, has intrigued researchers for years. Domination in graphs has a large following, and continues to be of interest to a wide variety of researchers both in the pure sciences and in applications. Haynes, Hedetniemi, and Slater ([3], [4]) brought together an immense amount of research in the area in the late 1990's, and have literally thousands of references on the topic between the two books. A set S of vertices from a graph G is a *dominating set* of G if and only if for every vertex $z \in V - S$, z is adjacent to a vertex in S. Merz et al. [9] took the concept of domination in graphs and defined the *domination graph* of a digraph D, $dom(D)$, to be the graph G where $V(G) = V(D)$ and xy is an edge of G if and only if for every $z \in V(D) - \{x, y\}$,

(x,z) or (y,z) is an arc in D. Thus, x and y form a directed dominating pair in D if and only if xy is an edge in $dom\,(D)$.

In 2008, Hedetniemi et al. [5] introduced the concept of $(1,k)$-dominating sets of a graph in general, and $(1,2)$-dominating sets in particular. A $(1,2)$-*dominating set* of a graph G is a set of vertices S such that for all vertices z in $V-S$, there is a vertex $x\in S$ where xz is an edge in G, and a second vertex $y\in S$ where y is at most distance 2 from z. The application motivating this research is modeled using a dominating set of vertices in a graph which represent guards at specific places in a building (vertices in the graph). In this case, we want one guard in S to be able to get to any unguarded vertex in one step and for a backup guard, also in S, to be no more than two steps away. Thus, we achieve the $(1,2)$-domination nature of the problem. Two additional papers follow the initial one ([6], [7]), where secondary and internal distances of sets in graphs are examined.

Just as historically the concept of domination graphs followed the concept of domination in graphs, so too the authors have followed the concept of $(1,k)$-domination in graphs with $(1,k)$-domination graphs [1]. Two vertices x and y in a digraph D are said to be a $(1,k)$-*dominating pair* if for every vertex $z\in V(D)-\{x,y\}$, either (x,z) or (y,z) is an arc in D, and there is a directed path of no more than k from the other of x and y to z. When $k=2$, this is known as *secondary domination.* The $(1,2)$-*domination graph of* D, $dom_{1,2}(D)$, has the same vertex set as D, with edge xy if and only if x and y form a $(1,2)$-dominating pair in D.

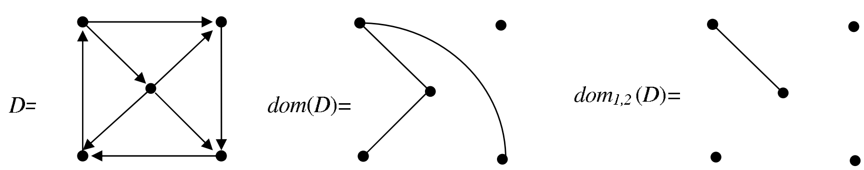

Figure 1: General digraph D with its domination and $(1,2)$-domination graphs.

The relationship between domination graphs and $(1,2)$-domination graphs is clear to see in Figure 1. Any edge in $dom_{1,2}(D)$ must have one dominating vertex with the added restriction that the second vertex be no more than two steps away. Thus, $dom_{1,2}(D)$ is a subgraph of $dom\,(D)$ [1].

In this paper, we look at the connected $(1,2)$-domination graphs of tournaments. Using connected domination graphs of tournaments as the pool in which to obtain our graphs, we are able to complete the characterization.

2. Connected $dom_{1,2}(T)$

When domination graphs were first introduced, the focus was on tournaments as the initial class of digraphs. Following that tradition, we begin by examining the $(1,2)$-

domination graphs of tournaments. A reasonable place to begin, with more than preliminaries, is trying to determine which $(1,2)$-domination graphs of tournaments are connected. Since we know that $dom_{1,2}(T)$ is a subgraph of $dom(T)$, it becomes extremely useful that the connected domination graphs of tournaments are well known.

Theorem 2.1. [10] *A connected graph is the domination graph of a tournament if and only if it is a spiked odd cycle, a star, or a caterpillar of positive length with three or more pendant vertices adjacent to one end of its spine.*

Using the results of Theorem 2.1 as a guide, we will explore which subgraphs of these domination graphs can be connected $(1,2)$-domination graphs.

2.1. Spiked Odd Cycles

For $dom_{1,2}(T)$ to be a connected subgraph of a spiked odd cycle with 0 or more spikes (pendant vertices), it must either equal the domination graph or be a connected proper subset. Consider the odd cycle with 0 pendant vertices, $dom(T) = C_n$, for n odd. There is only one tournament within isomorphic labeling for which this occurs, U_n. To define U_n, we let $n \geq 3$ be an odd integer. Form the regular tournament $T(S)$ where $S = \{1, 3, ..., n-2\}$ with arcs (i, j) if $j - i \equiv s \in S \pmod{n}$ and (j, i) otherwise. Then $T(S) = U_n$.

Theorem 2.2. [9] *Let T be a tournament on n odd vertices. Then $dom(T) = C_n$ if and only if $T \cong U_n$.*

Noting that U_n is a regular tournament and that $dom(U_n) = C_n$, we find that $dom_{1,2}(U_n)$ is a connected graph.

Theorem 2.3. [1] *Let T be a regular tournament. Then $dom_{1,2}(T) = dom(T)$.*

Corollary 2.4. *If $T = U_n$, then $dom_{1,2}(T) = C_n$ and is connected.*

As a preliminary to discussing a $(1,2)$-dominating graph that is a spiked odd cycle with at least one pendant vertex, we bring in the concept of a king. In a tournament, a vertex v is a *king* if v can reach every other vertex in 1 or 2 steps. With respect to secondary domination, kings are very important, as shown in the following theorem.

Theorem 2.5. [1] *Let T be a tournament with no dominating vertex. The pair u, v is a $(1,2)$-dominating pair if and only if u and v are a dominating pair of kings in T.*

Subtournaments where every vertex is a king play a large part in our characterization of the tournaments with connected $(1,2)$-domination graphs from this point forward. It is important to note that in a connected graph, every vertex is incident with an edge. Thus, when $dom_{1,2}(T)$ is connected, every vertex must be a king.

In the case of an all king n-tournament, it is not requisite for the vertices to also dominate, so our relationship is a bit different from the one where we have domination. The following lemma uses the insets of vertices to characterize vertices that are kings. We define the *inset of vertex v* as $O^-(v) = \{x \in V(T) \mid (x, v) \text{ is an arc in } T\}$.

Lemma 2.6. *Let T be a tournament on n vertices. A vertex $u \in V(T)$ is a king if and only if for all $v \in V(T) - \{u\}$, $O^-(v) \nsubseteq O^-(u)$.*

Proof. ($\Longrightarrow$) Consider $v \in V(T) - \{u\}$. Either (u, v) is an arc in T, or there is a vertex $y \in V(T)$ such that u, y, v is a uv-path. So, u or y is in $O^-(v)$ and not in $O^-(u)$. Thus, $O^-(v) \nsubseteq O^-(u)$.

($\Longleftarrow$) Consider $v \in V(T) - \{u\}$. Since $O^-(v) \nsubseteq O^-(u)$, either $u \in O^-(v)$ or there is a distinct vertex y such that $y \in O^-(v)$ and $y \notin O^-(u)$. This implies that either (u, v) is an arc in T, or u, y, v is a path in T. Thus, u is a king in T. $\square$

Corollary 2.7. *Let T be a tournament on n vertices. Every vertex of T is a king if and only if for all distinct $u, v \in V(T)$, $O^-(v) \nsubseteq O^-(u)$.*

Lemma 2.6 gives us a very important result regarding what must occur in T so that an edge of $dom(T)$ is not an edge in $dom_{1,2}(T)$. We define the distance between vertices u and v, $d(u, v)$, to be the length of the shortest directed path from u to v.

Corollary 2.8. *Let T be a tournament and uv an edge in $dom(T)$. Edge uv is not in $dom_{1,2}(T)$ if and only if there exists a vertex $z \in V - \{u, v\}$ such that $O^-(z) \subset O^-(u)$ or $O^-(z) \subset O^-(v)$.*

Proof. Since uv is an edge in $dom(T)$, u or v dominates z for all $z \in V(T) - \{u, v\}$. Edge uv will not be in $dom_{1,2}(T)$ if and only if for u or v there exists a vertex $z \in V(T) - \{u, v\}$ such that $d(u, z) \geq 3$ or $d(v, z) \geq 3$, indicating that u or v is not a king in T. From Lemma 2.6, this happens if and only if $O^-(z) \subset O^-(u)$ or $O^-(z) \subset O^-(v)$. $\square$

Now consider the possibility that $dom(T)$ is a spiked odd cycle with at least one pendant vertex. Fortunately, there are only certain tournaments that yield a domination graph that is a spiked odd cycle. In [9], Merz et al. described how to construct a tournament to realize any spiked odd cycle as a domination graph. In [8], Jimenez and Lundgren proved that this is the only type of tournament for which a spiked odd cycle is the domination graph. We shall refer to this tournament as a *spiked cycle tournament* in keeping with their terminology. Following is a description of this tournament class.

Let $x_0, ..., x_{k-1}$ be vertices forming U_k in T such that (x_i, x_{i+1}) is an arc in the tournament for $i = 1, ..., k-2$ and (x_{k-1}, x_0) is an arc, so that it is isomorphic to the labeling described for U_n. For each x_i let V_i be the set of pendant vertices adjacent to x_i. A set V_i can be empty. We define the *domination digraph of* T, $D(T)$, to be the digraph with underlying graph $dom(T)$, and arc (u, v) if uv is an edge in $dom(T)$ and u beats v in T. The general domination digraph for a spiked cycle tournament is given in Figure 2(a).

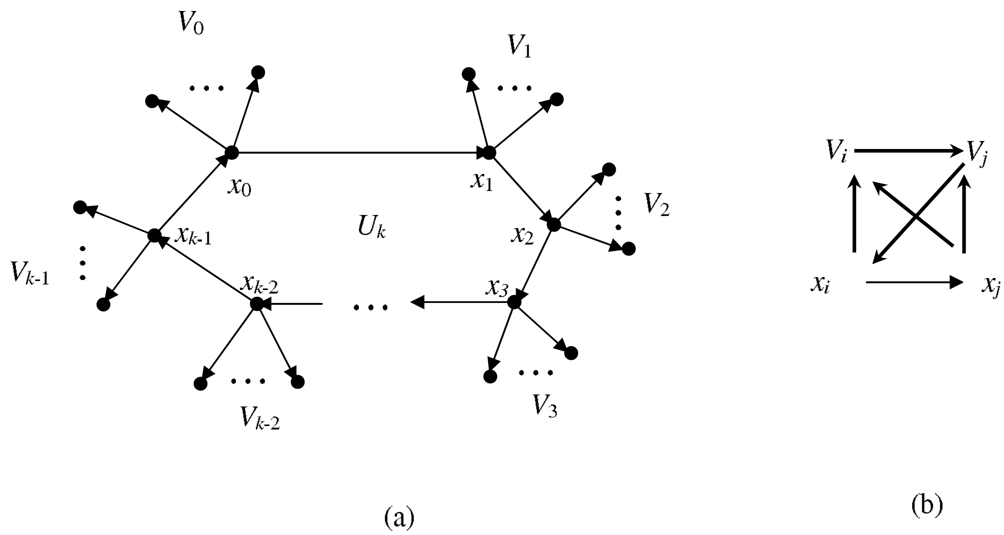

Figure 2: (a) The domination digraph, $D(T)$, of a spiked cycle tournament. (b) The mandated orientation of the arcs between the x_i and V_j of the spiked cycle tournament. Bold arcs represent arcs to/from all vertices in the V_m.

The remaining arcs in the spiked cycle tournament must be directed as shown in Figure 2(b). Note that (x_i, x_j) is an arc in T for each of the i, j pairs. The arcs within each V_i can be oriented arbitrarily.

Using the previous few results, we are now ready to address the subject of whether the $(1,2)$-domination graph of any spiked cycle tournament with at least one pendant vertex is a connected graph.

Theorem 2.9. *If T is a spiked cycle tournament with 1 or more pendant vertices, then $dom_{1,2}(T) \neq dom(T)$. Furthermore, $dom_{1,2}(T)$ is not connected.*

Proof. Let (x_i, v_i) be any pendant arc of T, and consider vertex x_{i+1} (mod n), in a spiked cycle tournament T. The arc (x_i, x_{i+1}) is in T, as are arcs (x_i, v_i) and (x_{i+1}, v_i) as seen in Figure 2(b). Our claim is that every vertex that beats x_{i+1} also beats v_i. Suppose that were not true. Then there is a vertex u such that (u, x_{i+1}) and (v_i, u) are both arcs. Consider $u = x_m$. Using Figure 2 as a guide, v_i only beats vertices x_m that beat x_i. Thus, (x_m, x_i) is an arc means $i - m \in \{1, 3, ..., n-2\}$ (mod n) according to the construction of U_k. But (x_m, x_{i+1}) is also an arc, implying that $(i+1) - m$ is also an element of $\{1, 3, ..., n-2\}$, which is impossible. Therefore, u cannot be one of the x_m on $V(U_k)$. So u must be a vertex v_m in some pendant set V_m. Vertex v_i only beats vertices $u = v_m$ where x_i beats x_m. So, (x_i, x_m) is an arc. Likewise, v_m only beats vertices on the cycle that beat x_m. So, (v_m, x_{i+1}) an arc in T mandates that (x_{i+1}, x_m) is an arc. By construction of U_k, this forces both $m - i$ and $m - (i+1)$ (mod n) to be in the set $\{1, 3, ..., n-2\}$. This is impossible, so no such vertex exists. Therefore, all vertices that beat x_{i+1} also beat v_i. Since x_{i+1} also beats v_i, $O^-(x_{i+1}) \subset O^-(v_i)$, and edge x_iv_i is not in $dom_{1,2}(T)$. Thus, $dom_{1,2}(T) \neq dom(T)$. Furthermore, since the edge x_iv_i is not in $dom_{1,2}(T)$, v_i is an isolated vertex and $dom_{1,2}(T)$ is not connected. □

2.2. Stars

The second possible connected domination graph that will be examined in terms of secondary domination is that of a star, $K_{1,n-1}$. For the discussion in this section, we need to identify more terms that will be used. Similar to the inset of a vertex v, the *outset of* v, $O^+(v)$, is the set of all vertices z such that (v,z) is an arc in D. The cardinality of the set $O^+(v)$ is denoted $d^+(v)$.

In [8], Jimenez and Lundgren show the two domination digraph structures of tournaments that yield a star as a domination graph. These are reproduced in Figure 3. Notice that the only difference is that the arc on the far left can go either way. When interested in the domination graph of these tournaments, the orientation of the arcs between vertices other than x is arbitrary. That is not the case when dealing with $(1,2)$-domination. Also, since a star is a tree, the removal of any edge disconnects the graph. For $dom_{1,2}(T)$ to be connected, it must therefore equal $dom(T)$.

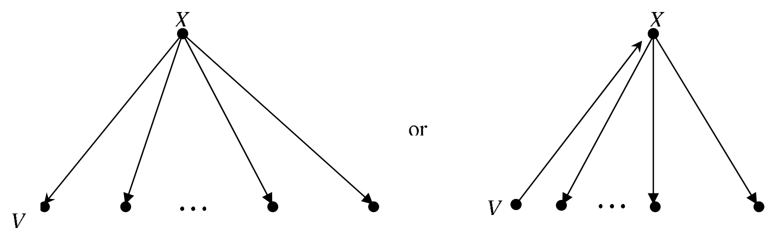

Figure 3: $D(T)$ of the only two possible tournaments with domination graphs of $K_{1,n-1}$.

Each $(1,2)$-dominating pair must follow the requirements of domination and secondary domination. Thus, it is imperative to see what must be true for the domination graph of the stellar tournament, and then apply the requirements for secondary domination.

Theorem 2.10. [8] *T is an n-tournament whose domination graph is a star, $K_{1,n-1}$, if and only if T is a tournament with a vertex x where $d^+(x) = n-1$ or a tournament with two distinguished vertices x and v such that i) (v,x) is an arc of T, and ii) for any $w_i \in V(T) - \{v,x\}$, (x,w_i) and (w_i,v) are arcs in T, and $T - \{x,v\}$ induces a subtournament where the indegree of each vertex is nonzero.*

By investigation of the two non-isomorphic tournaments on 3 vertices, one can determine that $K_{1,2}$ is impossible to achieve as a $(1,2)$-domination graph. Thus, we will consider $n \geq 4$.

Theorem 2.11. *Let $K_{1,n-1}$ for $n \geq 4$ be the domination graph of a tournament T. Then $dom_{1,2}(T) = dom(T)$ if and only if one of the following is true.*

1. *T has a vertex x where $d^+(x) = n-1$ and for the remaining $n-1$ vertices, $u_2, ..., u_n$, $O^-(u_i) \nsubseteq O^-(u_j)$, for all distinct $i, j = 2, ..., n$, or*
2. *T has vertices x and v such that $d^+(x) = n-2$, (v, x) is an arc in T, and*
   - *(a) (w_i, v) is an arc in T for every $w_i \in V(T) - \{v, x\}$, and*
   - *(b) for all distinct w_i, w_j, $O^-(w_i) \nsubseteq O^-(w_j)$.*

Proof. Let x be the center vertex of $K_{1,n-1}$ with pendant vertices v and w_i, $i = 1, ..., n-2$. The dominating vertices in each $(1,2)$-dominating pair must be of the form set forth in Theorem 2.10. What remains is to prove the structure of the secondary vertices.

($\Longrightarrow$) If $d^+(x) = n-1$ and $dom_{1,2}(T) = dom(T)$, then x is the dominating vertex of each $(1,2)$-dominating pair and we must consider the secondary vertices $u_2, ..., u_n$. These vertices must reach all vertices except x in at most 2 steps. Thus, they must all be kings in the subtournament induced on $V(T) - \{x\}$. From Lemma 2.6, they must have unique insets in the subtournament, so $O^-(u_i) \nsubseteq O^-(u_j)$ in the subtournament, and thus the tournament itself.

If (v, x) is an arc in T, then Theorem 2.10 also stipulates that for $i = 1, ..., n-2$, (w_i, v) must be an arc in T with nonzero indegree in the subtournament induced on the w_i. Since xv is an edge in $dom_{1,2}(T)$, they form a $(1,2)$-dominating pair, and v must beat the w_i in at most 2 steps. The path v, x, w_i accomplishes this, without modification to the tournament structure for the domination graph. The edges xw_i in $dom_{1,2}(T)$ are formed with the w_i dominating v, so we must have x reaching v within 2 steps for secondary domination. The path x, w_i, v for any one of the w_i is an xv-path of length 2, without modification to the tournament structure for the domination graph. Vertices w_i are the secondary vertices associated with the edges xw_i in $dom_{1,2}(T)$, so w_i must reach w_j within 2 steps for all distinct $i, j = 1, ..., n-2$. This forces the w_i to be kings in the subtournament generated on $V(T) - \{v, x\}$ in order to be secondary vertices. Using the same reasoning as the previous paragraph, $O^-(w_i) \nsubseteq O^-(w_j)$ for all distinct $i, j = 1, ..., n-2$.

($\Longleftarrow$) The formulation of kings in the theorem guarantees that every dominating pair in $dom(T)$ is also a $(1,2)$-dominating pair. Since $dom_{1,2}(T)$ is a subgraph of $dom(T)$, we have $dom_{1,2}(T) = dom(T) = K_{1,n-1}$. □

2.3. Caterpillars

A *caterpillar* is a tree where the removal of all pendant vertices results in a path. As in the case of a star, the removal of any edge from a caterpillar disconnects the graph. Thus, when examining connected $(1,2)$-domination graphs, we are only interested in tournaments where $dom_{1,2}(T) = dom(T)$ when $dom(T)$ is a caterpillar of positive length with 3 or more pendant vertices adjacent to one end of its spine.

In [2], Guichard et al. describe conditions for a tournament T such that $dom(T)$ is a caterpillar. Jimenez and Lundgren then show there is only one type of tournament for which this is true, and refer to it as a caterpillar tournament [8]. Actually, the caterpillar tournament is extremely similar to the spiked cycle tournament, blended with the stellar tournament from the previous subsection. We approach this subsection by using the results from [2] and [8] to motivate the results. Note that this may make the journey a bit longer, but it folds the history into the present result.

Before describing the caterpillar tournament here, we must first define a near-regular tournament. It is similar to U_n for n odd, as it follows a similar arc construction, but n is even. We let $n \geq 4$ be an even integer. Form the *near-regular tournament* U_n^* where vertices $x_0, ..., x_{n-2}$ induce U_{n-1}. For $i = 0, ..., n-2$, create arc (x_i, x_{n-1}) if $(n-1) - i$ is odd, and arc (x_{n-1}, x_i) otherwise. Note that the subtournament induced on vertices $x_1, ..., x_{n-1}$ also induces subtournament U_{n-1}.

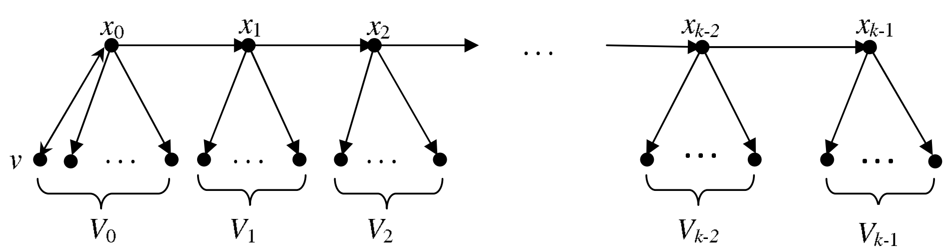

Figure 4: The $D(T)$ of the only possible caterpillar tournaments. Note that the arc from vertex v can go either way in T to obtain $dom(T)$.

Figure 4 shows the domination digraph for every tournament T where $dom(T)$ is a caterpillar of positive length with a cluster of at least 3 pendant vertices at one end. Note that either (v, x_0) or (x_0, v) can be an arc in the tournament. However, it turns out that if v exists and (v, x_0) is an arc, then k must be odd. In this case, we can relabel vertices with $x_i = x_{i+1}$ and $v = x_0$ so that k is even. Thus, we assume that k is even.

Tournament T is a *caterpillar tournament* if:

1. The vertex set can be partitioned into a "spine" set $\{x_0, ..., x_{k-1}\}$, $k \geq 2$ even, with possibly empty sets V_i of vertices pendant to x_i, for $i \neq k-1$,

2. The tournament induced on vertices $x_0, ..., x_{k-1}$ is U_k^*,

3. Arcs within V_i for $i = 0, ..., k-2$ are oriented arbitrarily,

4. There are at least 3 vertices within V_{k-1} and they induce a subtournament where all vertices have nonzero indegree,

5. The orientation of the arcs between x_i and V_j, and between V_i and V_j follows the pattern set forth in Figure 2(b), and

6. The remaining arcs are oriented as prescribed in Figure 4, except that vertex v will not be directed toward x_0.

The main theorem related to the caterpillar tournament follows.

Theorem 2.12. [8] *T is a tournament whose domination graph is a non-stellar caterpillar with a triple end if and only if T is a caterpillar tournament.*

Notice how similar $D(T)$ is for the spiked cycle tournament in Figure 2 and for the caterpillar tournament in Figure 4. The main difference (other than one being a cycle and the other a path) is that k is odd in the first instance and even in the second. In the definition of a caterpillar tournament, vertices $x_0, ..., x_{k-1}$ must induce U_k^*. That means vertices $x_0, ..., x_{k-2}$ and vertices $x_1, ..., x_{k-1}$ induce U_{k-1}. That is extremely close to the construction of the spiked odd cycle.

In Theorem 2.9, we found that there was no case where the pendant edges on the cycle in $dom(T)$ were also in $dom_{1,2}(T)$. What could change that here? The short answer is that there can be no pendant vertices on $x_0, ..., x_{k-2}$, as there is no way to change the relationship between x_{i+1} and $v_i \in V_i$ so that $O^-(x_{i+1}) \nsubseteq O^-(v_i)$.

Lemma 2.13. *Let T be a caterpillar tournament with spine vertices $x_0, ..., x_{k-1}$ and associated pendant vertex sets $V_0, ..., V_{k-1}$ for even $k \geq 2$. If $dom_{1,2}(T) = dom(T)$, then there are no pendant vertices adjacent to $x_0, ..., x_{k-2}$.*

Proof. Vertices $\{x_0, ..., x_{k-2}\} \cup \{V_0, ..., V_{k-2}\}$ generate a spiked odd cycle subtournament. In the subtournament, each edge $v_i x_i$ where $v_i \in V_i$ does not appear in the $(1,2)$-domination graph of T so will not be in any tournament where $dom_{1,2}(T) = dom(T)$. In the proof of Theorem 2.9, we found that $O^-(x_{i+1}) \subset O^-(v_i)$. We will show that this is still the case in the caterpillar tournament T by examining the arcs incident with the additional vertices x_{k-1} and $v \in V_{k-1}$. We look at two possibilities. The first is where (x_{k-1}, x_i) is an arc, and the second is where (x_i, x_{k-1}) is an arc. Since these dictate the direction of the vertices in V_{k-1}, all possible cases are considered. By construction, if (x_{k-1}, x_i) is an arc, then (x_{i+1}, x_{k-1}), (v, x_{i+1}) and (v, v_i) are arcs for $v_i \in V_i$, so $O^-(x_{i+1}) \subset O^-(v_i)$. If (x_i, x_{k-1}) is an arc, then (x_{k-1}, x_{i+1}) and (x_{k-1}, v_i) are arcs, so $O^-(x_{i+1}) \subset O^-(v_i)$. So, $O^-(x_{i+1}) \subset O^-(v_i)$ in all cases, and $v_i x_i$ is not an edge in $dom_{1,2}(T)$ for $v_i \in V_i$, $i = 1, ..., k-2$. Thus, there are no pendant vertices adjacent to $x_0, ..., x_{k-2}$. □

Remark 2.14. *If $k = 2$ it follows from this lemma that we have the case of a star. Thus, we will state our results for $k \geq 4$, which is the first nonstellar tournament for even k.*

The absence of pendant vertices on the first $k-1$ vertices of the spine certainly reduces the possible caterpillar $(1, 2)$-domination graphs. It remains to show that a path with an even number of vertices containing a cluster of at least 3 pendant vertices at one end is the $(1, 2)$-domination graph of some caterpillar tournament.

Theorem 2.15. *Let T be a caterpillar tournament with spine vertices $x_0, ..., x_{k-1}$ and associated pendant vertex sets $V_0, ..., V_{k-1}$ for even $k \geq 4$. Then $dom_{1,2}(T) = dom(T)$ if and only if $V_0, ..., V_{k-2}$ are empty sets, $|V_{k-1}| \geq 3$, and for all distinct $v_i, v_j \in V_{k-1}$, $O^-(v_i) \nsubseteq O^-(v_j)$.*

Proof. For constructions in this proof, refer to Figures 2(b) and 4. The dominating pairs must be of the form set forth in the definition of a caterpillar tournament. What remains is to prove the structure of the secondary vertices.

($\Longleftarrow$) We must show that each vertex is a king in T. First consider the x_i. The vertices $x_0, ..., x_{k-2}$ and vertices $x_1, ..., x_{k-1}$ each induce the subtournament U_{k-1}, which is a regular tournament. By Theorem 2.3, we know that each of the vertices within the two sets is a king within that set. Also, since (x_0, x_{k-1}) is an arc in T, and x_{k-1}, v, x_0 is a path for any $v \in V_{k-1}$, all of the vertices $x_0, ..., x_{k-1}$ are $(1, 2)$-dominating pairs in the subtournament generated on those $k-1$ vertices.

By construction, $\{x_0, x_2, ..., x_{k-2}\} \subset O^+(v)$ and $\{x_1, x_3, ..., x_{k-1}\} \subset O^-(v)$ for all $v \in V_{k-1}$. Using this information, we see that either (x_i, v) is an arc, or x_i, x_{i+1}, v is a path for all $v \in V_{k-1}$. Thus, the x_i are kings of T.

For the $v_i \in V_{k-1}$, consider that every x_j that beats one of the v_i beats all of the vertices in V_{k-1}. Therefore, the only difference in the insets of the v_i comes from the arcs in the subtournament induced on V_{k-1}. We are given that $O^-(v_i) \nsubseteq O^-(v_j)$. By Lemma 2.6, all of the v_i are kings in the subtournament induced by V_{k-1}. It remains to show that they reach all of the x_i in at most 2 steps. Similar to the previous argument, either (v, x_i) is an arc or v, x_{i-1}, x_i is a path. Thus, v is a king of T for all $v \in V_{k-1}$. Since every dominating pair is a $(1, 2)$-dominating pair, $dom_{1,2}(T) = dom(T)$.

($\Longrightarrow$) From Lemma 2.13, we see that $V_0, ..., V_{k-1}$ must be empty. This leaves the pendant vertex set V_{k-1} as the one that must have at least 3 vertices in order to be a caterpillar tournament. Since $dom_{1,2}(T) = dom(T)$, every $v \in V_{k-1}$ must be a king. The argument in the converse part of the proof shows that the vertices of V_{k-1} reach the x_i in at most 2 steps. They must also reach each other within two steps, so must be kings in the subtournament induced by V_{k-1}. Thus, $O^-(v_i) \nsubseteq O^-(v_j)$ for all unique $v_i, v_j \in V_{k-1}$. □

2.4. Conclusion

This short subsection provides a place to focus all of the results into one area. We characterize the tournaments whose $(1, 2)$-domination graphs are connected, and then give the list of graphs themselves.

Theorem 2.16. *Let T be a tournament on n vertices. Then $dom_{1,2}(T)$ is a connected graph if and only if one of the following is true:*

1. *T is isomorphic to U_n for odd $n \geq 3$, or*
2. *T has a vertex x where $d^+(x) = n-1$ and for the remaining $n-1$ vertices, $u_2, ..., u_n$, $O^-(u_i) \nsubseteq O^-(u_j)$, for all distinct $i, j = 2, ..., n$, $n \geq 4$, or*
3. *T has vertices x and v such that $d^+(x) = n-2$, (v, x) is an arc in T, $n \geq 4$, and*
   - *(a) (w_i, v) is an arc in T for every $w_i \in V(T) - \{v, x\}$, and*
   - *(b) for all distinct w_i, w_j, $O^-(w_i) \nsubseteq O^-(w_j)$, or*
4. *T is a caterpillar tournament where there are at least $k = 4$ even vertices in the spine, pendant vertex sets $V_0, ..., V_{k-2}$ are all empty, $|V_{k-1}| \geq 3$, and for all distinct $v_i, v_j \in V_{k-1}$, $O^-(v_i) \nsubseteq O^-(v_j)$, or*
5. *T is a tournament on 1 or 2 vertices.*

Proof. We know that $dom_{1,2}(T)$ is a subgraph of $dom(T)$. Thus, for $dom_{1,2}(T)$ to be connected, $dom(T)$ must be connected. Theorem 2.1 gives the three types of graphs that are connected domination graphs, so they are the only ones we need examine to find the connected $(1,2)$-domination graphs. Corollary 2.4 and Theorems 2.9, 2.11, and 2.15 give necessary and sufficient conditions for this to occur as well as detail when it cannot occur. They give the conditions listed in parts (1)-(4) of the theorem. If T is a tournament on 1 or 2 vertices then $dom_{1,2}(T) = UG(T)$, which is K_1 and K_2 respectively. Since there are no other domination graphs that are connected, the list is complete. □

Corollary 2.17. *Let T be a tournament on n vertices. Then $dom_{1,2}(T)$ is a connected graph if and only if $dom_{1,2}(T)$ is one of the following:*

1. *An odd cycle on $n \geq 3$ vertices, or*
2. *A star, $K_{1,n-1}$ on $n \geq 4$ vertices, or*
3. *A path on $n \geq 4$ even vertices with a minimum of 3 pendant vertices appended to one end, or*
4. *K_1 or K_2 for $n = 1$ and $n = 2$ respectively.*

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