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A Hierarchy of Maps Between Compacta

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Abstract. Let \mathbf{CH} be the class of compacta (i.e., compact Hausdorff spaces), with \mathbf{BS} the subclass of Boolean spaces. For each ordinal α and pair $\langle \mathbf{K}, \mathbf{L} \rangle$ of subclasses of \mathbf{CH} , we define $\text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{L})$, the class of maps of level at least α from spaces in \mathbf{K} to spaces in \mathbf{L} , in such a way that, for finite α , $\text{Lev}_{\geq \alpha}(\mathbf{BS}, \mathbf{BS})$ consists of the Stone duals of Boolean lattice embeddings that preserve all prenex first-order formulas of quantifier rank α . Maps of level ≥ 0 are just the continuous surjections, and the maps of level ≥ 1 are the co-existential maps introduced in [8]. Co-elementary maps are of level $\geq \alpha$ for all ordinals α : of course in the Boolean context, the co-elementary maps coincide with the maps of level $\geq \omega$. The results of this paper include:

- (i) every map of level $\geq \omega$ is co-elementary;
- (ii) the limit maps of an ω -indexed inverse system of maps of level $\geq \alpha$ are also of level $\geq \alpha$; and
- (iii) if \mathbf{K} is a co-elementary class, $k < \omega$ and $\text{Lev}_{\geq k}(\mathbf{K}, \mathbf{K}) = \text{Lev}_{\geq k+1}(\mathbf{K}, \mathbf{K})$, then $\text{Lev}_{\geq k}(\mathbf{K}, \mathbf{K}) = \text{Lev}_{\geq \omega}(\mathbf{K}, \mathbf{K})$.

A space $X \in \mathbf{K}$ is *co-existentially closed* in \mathbf{K} if $\text{Lev}_{\geq 0}(\mathbf{K}, X) = \text{Lev}_{\geq 1}(\mathbf{K}, X)$. Adapting the technique of “adding roots,” by which one builds algebraically closed extensions of fields (and, more generally, existentially closed extensions of models of universal-existential theories), we showed in [8] that every infinite member of a co-inductive co-elementary class (such as \mathbf{CH} itself, \mathbf{BS} , or the class \mathbf{CON} of continua) is a continuous image of a space of the same weight that is co-existentially closed in that class. We show here that every compactum that is co-existentially closed in \mathbf{CON} (a *co-existentially closed continuum*) is both indecomposable and of covering dimension one.

§1. Introduction. This paper, a continuation of [2]–[8], aims to carry on the project of establishing model-theoretic concepts and methods within the topological context; namely that of compacta (i.e., compact Hausdorff spaces). Since there is a precise duality between the categories of compacta (plus continuous maps) and commutative B^* -algebras (plus nonexpansive linear maps) (the Gel’fand-Naïmark theorem [21]), our enterprise may also be seen as part of Banach model theory (see [12]–[16]). The main difference is that we are doing Banach model theory “in the mirror,” so to speak, and it is often the case that a mirror can help one focus on features that might otherwise go unnoticed. The paragraph introducing Theorem 4.5 below illustrates this. The algebraic/model-theoretic technique of “adding roots” in order to produce algebraically closed extensions of fields (and, more generally, existentially closed extensions of models of a universal-existential theory)

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may be used to produce new kinds of compacta. Some of these, the *co-existentially closed continua*, are indecomposable and unidimensional. It is difficult to see how this result would have been possible, had we been focusing on commutative B^* -algebras.

In the interests of being as self-contained as possible, we present a quick review of our main tool, the topological ultraproduct construction. It is this construction, plus the landmark ultrapower theorem of Keisler-Shelah [9], that gets our project off the ground. (Detailed accounts may be found in [2]–[6] and [11].)

We let \mathbf{CH} denote the category of compacta and continuous maps. In model theory, it is well known that ultraproducts may be described in the language of category theory; i.e., as direct limits of (cartesian) products, where the directed set is the ultrafilter with reverse inclusion, and the system of products consists of cartesian products taken over the various sets in the ultrafilter. (Bonding maps are just the obvious restriction maps.) When we transport this framework to the category-opposite of \mathbf{CH} , the result is the *topological ultraproduct* (i.e., take an inverse limit of coproducts), and may be concretely described as follows: Given compacta $\langle X_i : i \in I \rangle$ and an ultrafilter \mathcal{D} on I , let Y be the disjoint union $\bigcup_{i \in I} (X_i \times \{i\})$ (a locally compact space). With $q : Y \rightarrow I$ the natural projection onto the second coördinate (where I has the discrete topology), we then have the Stone-Čech lifting $q^\beta : \beta(Y) \rightarrow \beta(I)$. Now the ultrafilter \mathcal{D} may be naturally viewed as an element of $\beta(I)$, and it is not hard to show that the topological ultraproduct $\sum_{\mathcal{D}} X_i$ is the pre-image $(q^\beta)^{-1}[\mathcal{D}]$. (The reader may be familiar with the Banach ultraproduct [10]. This construction is indeed the ultraproduct in the category of Banach spaces and nonexpansive linear maps, and may be telegraphically described using the recipe: take the usual ultraproduct, throw away the infinite elements, and mod out by the subspace of infinitesimals. Letting $C(X)$ denote the Banach space of continuous real-valued (or complex-valued) continuous functions with X as domain, the Banach ultraproduct of $\langle C(X_i) : i \in I \rangle$ via \mathcal{D} is just $C(\sum_{\mathcal{D}} X_i)$.)

If $X_i = X$ for all $i \in I$, then we have the *topological ultrapower* $XI \setminus \mathcal{D}$, a subspace of $\beta(X \times I)$. In this case there is the Stone-Čech lifting p^β of the natural first-coördinate map $p : X \times I \rightarrow X$. Its restriction to the ultrapower is a continuous surjection, called the *codiagonal map*, and is officially denoted $p_{X, \mathcal{D}}$ (with the occasional notation-shortening alias possible). This map is dual to the natural diagonal map from a relational structure to an ultrapower of that structure, and is not unlike the standard part map from nonstandard analysis.)

When attention is restricted to the full subcategory \mathbf{BS} of Boolean spaces, Stone duality assures us that the ultraproduct construction matches perfectly with the usual ultraproduct construction for Boolean lattices. This says that “dualized model theory” in \mathbf{BS} is largely a predictable rephrasing of the usual model theory of the elementary class of Boolean lattices. In the category \mathbf{CH} , however, there is no similar match (see [1] and [19]); one is forced to look for other (less direct) model-theoretic aids. Fortunately there is a finitely axiomatizable AE Horn class of bounded distributive lattices, the so-called *normal disjunctive* lattices [6] (also called Wallman lattices in [5]), comprising precisely the (isomorphic copies of) *lattice bases*, those lattices that serve as bases for the closed sets of compacta. (To be more specific: The normal disjunctive lattices are precisely those bounded lattices A such that there exists a compactum X and a meet-dense sublattice \mathcal{A} of

the closed set lattice $F(X)$ of X such that A is isomorphic to \mathcal{A} .) We go from lattices to spaces, as in the case of Stone duality, via the *maximal spectrum* $S(\)$, pioneered by H. Wallman [23]. $S(A)$ is the space of maximal proper filters of A ; a typical basic closed set in $S(A)$ is the set a^\sharp of elements of $S(A)$ containing a given element $a \in A$. $S(A)$ is generally compact with this topology. Normality, the condition that if a and b are disjoint ($a \sqcap b = \perp$), then there are a', b' such that $a \sqcap a' = b \sqcap b' = \perp$ and $a' \sqcup b' = \top$, ensures that the maximal spectrum topology is Hausdorff. Disjunctivity, which says that for any two distinct lattice elements there is a nonbottom element that is below one and disjoint from the other, ensures that the map $a \mapsto a^\sharp$ takes A isomorphically onto the canonical closed set base for $S(A)$. $S(\)$ is contravariantly functorial: If $f: A \rightarrow B$ is a homomorphism of normal disjunctive lattices and $M \in S(B)$, then $f^S(M)$ is the unique maximal filter extending the prime filter $f^{-1}[M]$. (For normal lattices, each prime filter is contained in a unique maximal one.)

The ultrapower theorem states that two relational structures are elementarily equivalent if and only if some ultrapower of one is isomorphic to some ultrapower of the other. One may easily extend this result, by the use of added constant symbols, to show that a function $f: A \rightarrow B$ between structures is an elementary embedding if and only if there is an isomorphism of ultrapowers $h: A^I/\mathcal{D} \rightarrow B^J/\mathcal{E}$ such that the obvious mapping square commutes; i.e., such that $d_{\mathcal{E}} \circ f = h \circ d_{\mathcal{D}}$, where $d_{\mathcal{D}}$ and $d_{\mathcal{E}}$ are the natural diagonal embeddings. This characterization is used, in a thoroughly straightforward way, to *define* what it means for two compacta to be *co-elementarily equivalent* and for a map between compacta to be a *co-elementary map*. It is a relatively easy task to show, then, that $S(\)$ converts ultraproducts to ultracoproducts, elementarily equivalent lattices to co-elementarily equivalent compacta, and elementary embeddings to co-elementary maps. Furthermore, if $f: A \rightarrow B$ is a *separative* embedding; i.e., an embedding such that if $b \sqcap c = \perp$ in B , then there exists $a \in A$ such that $f(a) \geq b$ and $f(a) \sqcap c = \perp$, then f^S is a homeomorphism (see [2], [4], [5], [6], and [11]). Because of this, there is much flexibility in how we may obtain $\sum_{\mathcal{D}} X_i$: Simply choose a lattice base \mathcal{A}_i for each X_i and apply $S(\)$ to the ultraproduct $\prod_{\mathcal{D}} \mathcal{A}_i$.

The spectrum functor falls far short of being a duality, except when restricted to the Boolean lattices. For this reason, one must take care not to jump to too many optimistic conclusions; such as inferring that if compacta X and Y are co-elementarily equivalent, then there must be lattice bases \mathcal{A} for X and \mathcal{B} for Y such that $\mathcal{A} \equiv \mathcal{B}$. Similarly, one may not assume that a co-elementary map is of the form f^S for some elementary embedding. This “representation problem” has yet to be solved.

§2. An ordinal-indexed hierarchy of maps. Recall the definition of quantifier rank for first-order formulas in prenex normal form: φ is of *rank* 0 if it is quantifier free; for $k < \omega$, φ is of *rank* $k + 1$ if φ is of the form $\pi\psi$, where ψ is a prenex formula of rank k , and π is a prefix of like quantifiers, of polarity opposite to that of the leading quantifier of ψ (if there is one). We use the notation $\varphi(x_1, \dots, x_n)$ to mean that the free variables occurring in φ come from the set $\{x_1, \dots, x_n\}$. For $k < \omega$, a function $f: A \rightarrow B$ between structures is a *map of level* $\geq k$ if for every formula $\varphi(x_1, \dots, x_n)$ of rank k and every n -tuple $\langle a_1, \dots, a_n \rangle$ from A , $A \models \varphi[a_1, \dots, a_n]$ (if and) only if

$B \models \varphi[f(a_1), \dots, f(a_n)]$. (The obvious substitution convention is being followed here.) Maps of level ≥ 0 are just the embeddings; maps of level ≥ 1 are often called *existential embeddings*. (So the image under an existential embedding of one structure into another is existentially closed in the larger structure.) Of course an embedding is elementary if and only if it is of level $\geq \omega$; i.e., of level $\geq k$ for all $k < \omega$.

The following result is well known (see [22]), and forms the basis upon which we can export the model-theoretic notion of *map of level k* to the topological context. A function $f: A \rightarrow B$ between relational structures is a map of level $\geq k + 1$ if and only if there are functions $g: A \rightarrow C$, $h: B \rightarrow C$ such that g is an elementary embedding, h is a map of level $\geq k$, and $g = h \circ f$. (C may be taken to be an ultrapower of A , with g the natural diagonal.)

We then define the notion of *map of level k* in the compact Hausdorff context by use of this characterization. $f: X \rightarrow Y$ is of level ≥ 0 if it is a continuous surjection; for $k < \omega$, f is of level $\geq k + 1$ if there are functions $g: Z \rightarrow Y$, $h: Z \rightarrow X$ such that g is a co-elementary map, h is a map of level $\geq k$, and $g = f \circ h$. If f happens to be of level $\geq k$ for every $k < \omega$, there is no obvious reason to infer that f is co-elementary. It therefore makes sense to carry the hierarchy into the transfinite, taking intersections at the limit stages and mimicking the inductive stage above otherwise. Thus we may talk of maps of level $\geq \alpha$ for α any ordinal. Clearly co-elementary maps are of level $\geq \alpha$ for each α , but indeed there is no obvious assurance that the converse is true. The main goal of this section is to show that being of level $\geq \omega$ is in fact equivalent to being co-elementary.

REMARKS 2.1.

(i) Co-elementary equivalence is known ([2], [5], and [11]) to preserve important properties of topological spaces, such as being infinite, being a continuum (i.e., connected), being Boolean (i.e., totally disconnected), having (Lebesgue covering) dimension n , and being a decomposable continuum. If $f: X \rightarrow Y$ is a co-elementary map in **CH**, then of course X and Y are co-elementarily equivalent ($X \equiv Y$). Moreover, since f is a continuous surjection (see [2]), additional information about X is transferred to Y . For instance, continuous surjections in **CH** cannot raise *weight* (i.e., the smallest cardinality of a possible topological base, and for many reasons the right cardinal invariant to replace cardinality in the dualized model-theoretic setting), so metrizable (i.e., being of countable weight in the compact Hausdorff context) is preserved. Also local connectedness is preserved, since continuous surjections in **CH** are quotient maps. Neither of these properties is an invariant of co-elementary equivalence alone.

(ii) A number of properties, not generally preserved by continuous surjections between compacta, are known [8] to be preserved by co-existential (i.e., level ≥ 1) maps. Among these are: being infinite, being disconnected, having dimension $\leq n$, and being an (hereditarily) indecomposable continuum.

As is shown in [2], co-elementary equivalence is an equivalence relation (the sticking point being transitivity, of course), and the composition of co-elementary maps is again a co-elementary map. Furthermore, there is the following “closure under terminal factors” property: If f and $g \circ f$ are co-elementary maps, then so is g . What makes these (and many other) results work is the following lemma, an

application of a strong form of Shelah’s version of the ultrapower theorem (see [8], [20]). The following is a slight rephrasing of Lemma 2.1 in [8], and is proved the same way.

LEMMA 2.2. *Let $\langle \langle X_\delta, f_\delta, Y_\delta \rangle : \delta \in \Delta \rangle$ be a family of triples, where f_δ either indicates co-elementary equivalence between X_δ and Y_δ , or is a co-elementary map from X_δ to Y_δ , both spaces being compacta. Then there is a single ultrafilter witness to the fact. More precisely, there is an ultrafilter \mathcal{D} on a set I and a family of homeomorphisms $\langle h_\delta : X_\delta I \setminus \mathcal{D} \rightarrow Y_\delta I \setminus \mathcal{D} : \delta \in \Delta \rangle$ such that $f_\delta \circ p_{X_\delta, \mathcal{D}} = p_{Y_\delta, \mathcal{D}} \circ h_\delta$ whenever f_δ is a co-elementary map.*

In order for us to prove any substantial results concerning maps of level $\geq \alpha$, we must extend the ultraproduct construction from compacta to continuous maps between compacta. This was originally done in [2], but we need to establish some new facts about this construction.

Recall that if $f_i : X_i \rightarrow Y_i$ is a continuous map for each $i \in I$, and \mathcal{D} is an ultrafilter on I , then $\sum_{\mathcal{D}} f_i : \sum_{\mathcal{D}} X_i \rightarrow \sum_{\mathcal{D}} Y_i$ may be defined as $(\prod_{\mathcal{D}} f_i^F)^S$, where f_i^F is just “pulling closed sets back to closed sets,” and the ultraproduct map at the lattice level is defined in the usual way. When all the maps f_i are equal to a single map f , we have the ultracopower map, which we denote $fI \setminus \mathcal{D}$. It is straightforward to show that ultraproducts of continuous surjections (resp., homeomorphisms) are again continuous surjections (resp., homeomorphisms). In particular the ultracopower operation $()I \setminus \mathcal{D}$ is an endofunctor on the category **CH**.

PROPOSITION 2.3. *Ultracoproduts of co-elementary maps are co-elementary maps. More specifically, if $\{ i \in I : f_i \text{ is a co-elementary map} \} \in \mathcal{D}$, then $\sum_{\mathcal{D}} f_i$ is a co-elementary map also.*

PROOF. Let $f_i : X_i \rightarrow Y_i$ be given, $i \in I$, and set

$$J := \{ i \in I : f_i \text{ is a co-elementary map} \} \in \mathcal{D}.$$

We first consider the special case where $X_i := Y_i K_i \setminus \mathcal{E}_i$, and $f_i := p_{\mathcal{E}_i}$ (the codiagonal map), for $i \in J$. Then $f_i = d_{\mathcal{E}_i}^S$, the image under the maximal spectrum functor of the canonical diagonal embedding taking $F(Y_i)$ to $F(Y_i)^{K_i} / \mathcal{E}_i$. Now each $d_{\mathcal{E}_i}$ is an elementary embedding; and an easy consequence of the Łoś ultraproduct theorem is that ultraproducts of elementary embeddings are elementary. Since $S()$ converts elementary embeddings to co-elementary maps, we conclude that $\sum_{\mathcal{D}} f_i$ is a co-elementary map in this case.

In general, we have, for $i \in J$, homeomorphisms between ultracopowers

$$h_i : X_i K_i \setminus \mathcal{D}_i \rightarrow Y_i L_i \setminus \mathcal{E}_i,$$

with $f_i \circ p_{\mathcal{D}_i} = p_{\mathcal{E}_i} \circ h_i$. When we take the ultracoprodut, commutativity is preserved, $\sum_{\mathcal{D}} h_i$ is a homeomorphism, and $\sum_{\mathcal{D}} p_{\mathcal{D}_i}$ and $\sum_{\mathcal{D}} p_{\mathcal{E}_i}$ are both co-elementary. Thus $\sum_{\mathcal{D}} f_i$ is co-elementary, by closure under terminal factors. \dashv

The following analogue of 2.3 can now be easily proved.

COROLLARY 2.4. *For each ordinal α , ultracoproduts of maps of level $\geq \alpha$ are maps of level $\geq \alpha$.*

PROOF. The proof is by induction on α . Ultraproducts of continuous surjections are continuous surjections, so the result is established for $\alpha = 0$. The inductive step at limit ordinals is trivial, so it remains to prove the inductive step at successor ordinals. But this follows immediately from 2.3 and the definition of being of level $\geq \alpha + 1$. \dashv

Next we need closure under composition.

PROPOSITION 2.5. *For each ordinal α , the composition of two maps of level $\geq \alpha$ is a map of level $\geq \alpha$.*

PROOF. Again we prove by induction on α . There is no problem for α either zero or a positive limit ordinal, so assume the composition of two maps of level $\geq \alpha$ is also of level $\geq \alpha$, and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be maps of level $\geq \alpha + 1$. By definition, there are maps $u: W \rightarrow X$, $v: W \rightarrow Y$ such that u is co-elementary, v is of level $\geq \alpha$, and $u = f \circ v$. By the co-elementarity of u , there are ultracopowers $p: YI \setminus \mathcal{D} \rightarrow Y$, $q: WJ \setminus \mathcal{E} \rightarrow W$, and a homeomorphism $h: WJ \setminus \mathcal{E} \rightarrow YI \setminus \mathcal{D}$ such that $u \circ q = p \circ h$. By our inductive hypothesis, $v \circ q \circ h^{-1}$ is of level $\geq \alpha$; so we are justified in assuming that the co-elementary part of a witness to a map's being of level $\geq \alpha$ may be taken to be an ultracopower codiagonal map.

Getting back to f and g , and using 2.2, there are maps $p: YI \setminus \mathcal{D} \rightarrow Y$, $h: YI \setminus \mathcal{D} \rightarrow X$, $q: ZI \setminus \mathcal{D} \rightarrow Z$, $j: ZI \setminus \mathcal{D} \rightarrow Y$ such that p and q are codiagonal maps, h and j are maps of level $\geq \alpha$, and the equalities $p = f \circ h$ and $q = g \circ j$ both hold. By 2.4, the ultracopower map $gI \setminus \mathcal{D}$ is of level $\geq \alpha + 1$. Thus we have further witnesses $u: W \rightarrow ZI \setminus \mathcal{D}$ and $v: W \rightarrow YI \setminus \mathcal{D}$ such that u is co-elementary, v is of level $\geq \alpha$, and $u = (gI \setminus \mathcal{D}) \circ v$. Now $h \circ v$ is of level $\geq \alpha$ by our inductive hypothesis, $q \circ v$ is co-elementary by the long-established fact [2] that co-elementarity is closed under composition, and it is a routine exercise to establish that $q \circ u = g \circ f \circ h \circ v$. Thus $g \circ f$ is of level $\geq \alpha + 1$. \dashv

COROLLARY 2.6. *Let α be an ordinal, $f: X \rightarrow Y$ a map of level $\geq \alpha + 1$. Then there is an ultracopower map $p: YI \setminus \mathcal{D} \rightarrow Y$ and a map $h: YI \setminus \mathcal{D} \rightarrow X$ of level $\geq \alpha$ such that $p = f \circ h$.*

PROOF. This is immediate from 2.5, plus the definitions of *co-elementary map* and *map of level $\alpha + 1$* . \dashv

2.5 gives us the following analogue of closure under terminal factors for co-elementary maps.

COROLLARY 2.7. *Let α be an ordinal. If h is of level $\geq \alpha$ and $f \circ h$ is of level $\geq \alpha + 1$, then f is of level $\geq \alpha + 1$.*

PROOF. Let $f: X \rightarrow Y$, $h: Z \rightarrow X$ be given, where h is of level $\geq \alpha$ and $g := f \circ h$ is of level $\geq \alpha + 1$. Then we have, as witness to the level of g , maps $u: W \rightarrow Y$ and $v: W \rightarrow Z$ such that u is co-elementary, v is of level $\geq \alpha$, and $u = g \circ v$. By 2.5, $h \circ v$ is of level $\geq \alpha$, and we have a witness to the fact that f is of level $\geq \alpha + 1$. \dashv

We may now establish a needed consequence of 2.3, 2.4, and 2.5.

PROPOSITION 2.8. *Let $f : X \rightarrow Y$ be a map of level $\geq \omega$ between compacta. Then there are maps $g : Z \rightarrow Y$ and $h : Z \rightarrow X$ such that g is co-elementary, h is of level $\geq \omega$, and $g = f \circ h$. Moreover, g may be taken to be an ultracopower codiagonal map.*

PROOF. For each $k < \omega$, f is of level $\geq k + 1$. So let $g_k : Z_k \rightarrow Y$ and $h_k : Z_k \rightarrow X$ witness the fact; each g_k is co-elementary, each h_k is of level $\geq k$, and $g_k = f \circ h_k$. Let \mathcal{D} be any nonprincipal ultrafilter on ω . For each $k < \omega$ we then have

$$\{k \in \omega : g_k \text{ is co-elementary and } h_k \text{ is of level } \geq k\} \in \mathcal{D}.$$

By 2.3, $\sum_{\mathcal{D}} g_k$ is co-elementary; by 2.4, $\sum_{\mathcal{D}} h_k$ is of level $\geq \omega$, and $\sum_{\mathcal{D}} g_k = (f \omega \setminus \mathcal{D}) \circ (\sum_{\mathcal{D}} h_k)$. Let $p : X \omega \setminus \mathcal{D} \rightarrow X$ and $q : Y \omega \setminus \mathcal{D} \rightarrow Y$ be the codiagonal maps. Then, by 2.5, $p \circ (\sum_{\mathcal{D}} h_k)$ is of level $\geq \omega$. We also have the co-elementarity of $q \circ (\sum_{\mathcal{D}} g_k)$, as well as the equality $f \circ p \circ (\sum_{\mathcal{D}} h_k) = q \circ (\sum_{\mathcal{D}} g_k)$; hence the desired result. Further application of 2.5 makes it possible to arrange for $g : Z \rightarrow Y$ to be an ultracopower codiagonal map. \dashv

In order to prove that maps of level $\geq \omega$ are co-elementary, we need a result on *co-elementary chains*. Suppose

$$\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$$

is an ω -indexed inverse system of maps between compacta. Then there is a compactum X and maps $g_n : X \rightarrow X_n$, $n < \omega$, such that the equalities $g_n = f_n \circ g_{n+1}$ all hold. Moreover, X is “universal” in the sense that if $h_n : Y \rightarrow X_n$ is any other family of maps such that the equations $h_n = f_n \circ h_{n+1}$ all hold, then there is a unique $f : Y \rightarrow X$ such that $h_n = g_n \circ f$ for all $n < \omega$. X is the *inverse limit* of the sequence, and may be described as the subspace

$$\{ \langle x_0, x_1, \dots \rangle \in \prod_{n < \omega} X_n : x_n = f_n(x_{n+1}) \text{ for each } n < \omega \}.$$

The limit map g_n is then just the projection onto the n th factor.

The inverse system is called a *co-elementary chain* if each f_n is a co-elementary map. We would like to conclude that, with co-elementary chains, the limit maps g_n are also co-elementary. This would give us a perfect analogue of the Tarski-Vaught elementary chains theorem (see [9]). In the model-theoretic version, the proof uses induction on the complexity of formulas, and is elegantly simple. In our setting, however, it is not entirely obvious how to proceed with a proof. The result is still true, but there is no simple elegant proof that we know of. One proof is outlined in [8] (see Theorem 4.2 there). It uses an elementary chains analogue in Banach model theory, plus the Gel’fand-Naïmark duality theorem. We present two more proofs in the next section; ones that use only the techniques we have developed so far.

An important step on the way to the co-elementary chains theorem is the result that every map of level $\geq \omega$ is co-elementary. It turns out that this step itself uses the co-elementary chains theorem, but only in a weak form. Given a co-elementary chain

$$\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle,$$

we say the chain is *representable* if there is an elementary chain

$$\langle A_n \xrightarrow{r_n} A_{n+1} : n < \omega \rangle$$

of normal disjunctive lattices such that, for each $n < \omega$, $X_n = S(A_n)$ and $f_n = r_n^S$.

LEMMA 2.9. *Let $\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$ be a representable co-elementary chain, with inverse limit X and limit maps $g_n : X \rightarrow X_n$, $n < \omega$. Then each g_n is a co-elementary map.*

PROOF. Let $\langle A_n \xrightarrow{r_n} A_{n+1} : n < \omega \rangle$ represent our co-elementary chain in the sense given above. Let A be the direct limit of this direct system, with limit maps $t_n : A_n \rightarrow A$. Then the Tarski-Vaught theorem says that each t_n is an elementary embedding. Since the maximal spectrum functor converts elementary embeddings to co-elementary maps, we then have $Y := S(A)$, and co-elementary maps $h_n := t_n^S$. Let $f : Y \rightarrow X$ be defined by the equalities $h_n = g_n \circ f$. Applying the closed set functor $F(\)$ to the representing elementary chain, letting $u_n : A_n \rightarrow F(X_n)$ be the natural separative embedding, we have the embeddings $g_n^F \circ u_n : A_n \rightarrow F(X)$. We then get $u : A \rightarrow F(X)$, defined by $u \circ t_n = g_n^F \circ u_n$. Let $g := u^S : X \rightarrow Y$. Then we have, applying $S(\)$, and noting that the maps u_n^S are canonical homeomorphisms, $u_n^S \circ g_n = h_n \circ g$. This implies that f and g are inverses of one another; hence that the maps g_n are co-elementary. -1

We are now ready to prove the main result of this section.

THEOREM 2.10. *Every map of level $\geq \omega$ is co-elementary.*

PROOF. Let $f_0 : X_0 \rightarrow Y_0$ be a map of level $\geq \omega$. We build a “co-elementary ladder” over this map as follows: By 2.8, there are maps $g_0 : Y_1 \rightarrow Y_0$ and $h_0 : Y_1 \rightarrow X_0$ such that g_0 is co-elementary, h_0 is of level $\geq \omega$, and $g_0 = f_0 \circ h_0$. Moreover, we may (and do) take g_0 to be an ultracopower codiagonal map. Since h_0 is of level $\geq \omega$, we have maps $j_0 : X_1 \rightarrow X_0$ and $f_1 : X_1 \rightarrow Y_1$ such that j_0 is co-elementary, f_1 is of level $\geq \omega$, and $j_0 = h_0 \circ f_1$. As before, we take j_0 to be an ultracopower codiagonal map. This completes the first “rung” of the ladder, and we repeat the process for the map $f_1 : X_1 \rightarrow Y_1$. In the end, we have two co-elementary chains

$$\langle X_n \xleftarrow{j_n} X_{n+1} : n < \omega \rangle \quad \text{and} \quad \langle Y_n \xleftarrow{g_n} Y_{n+1} : n < \omega \rangle,$$

with inverse limits X and Y respectively. For each $n < \omega$, let $v_n : X \rightarrow X_n$ and $w_n : Y \rightarrow Y_n$ be the limit maps, defined by the equalities $v_n = j_n \circ v_{n+1}$, $w_n = g_n \circ w_{n+1}$.

Now each successive entry is an ultracopower of the last; hence these co-elementary chains are representable (by elementary chains of iterated ultrapowers). By 2.9, the maps v_n and w_n are co-elementary.

Consider now the maps $h_n : Y_{n+1} \rightarrow X_n$, $n < \omega$. These, along with the maps f_n , give rise to the existence of maps $f : X \rightarrow Y$ and $h : Y \rightarrow X$ that are unique with the property that for all $n < \omega$, $w_n \circ f = f_n \circ v_n$ and $v_n \circ h = h_n \circ w_{n+1}$. The uniqueness feature ensures that f and h are inverses of one another; thus f_0 is co-elementary, by closure under terminal factors. -1

§3. Inverse limits of α -chains. In this section we prove the co-elementary chains theorem in two different ways, both of which use 2.10.

If α is an ordinal, an inverse system $\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$ of maps between compacta is an α -chain if each f_n is a map of level $\geq \alpha$. By the α -chains theorem, we mean the statement that the limit maps of every α -chain are maps of level $\geq \alpha$. (So, for example, the 0-chains theorem is a well-known exercise.) Because of 2.10, the co-elementary chains theorem is just the ω -chains theorem; and this case clearly follows from the conjunction of the cases $\alpha < \omega$. While we do ultimately prove the α -chains theorem for general α , we first take a slight detour and establish the $\alpha = \omega$ case separately. The main reason for doing this (aside from the fact that we discovered this case first in an abortive attempt to establish the general case) is that it uses the following result, which is of further use later on, as well as being of some independent interest.

LEMMA 3.1. *Let $f : X \rightarrow Y$ be a function between compacta, let α be an ordinal, and let \mathcal{B} be a lattice base for Y . Suppose that for each finite $\delta \subseteq \mathcal{B}$ there is a map $g_\delta : Y \rightarrow Z_\delta$, of level $\geq \alpha$, such that $g_\delta \circ f$ is of level $\geq \alpha$ and for each $B \in \delta$, $g_\delta^{-1}[g_\delta[B]] = B$ (i.e., B is g_δ -saturated). Then f is a map of level $\geq \alpha$.*

PROOF. The proof below uses the basic idea for proving Theorem 3.3 in [8].

For each ordinal α , let \mathbf{A}_α be the assertion of the lemma for maps of level $\geq \alpha$. Then \mathbf{A}_ω follows immediately from the conjunction of the assertions \mathbf{A}_α for α finite. In view of 2.10, then, we may focus our attention on the finite case. While our proof is not by induction, it does require a separate argument for the case $\alpha = 0$.

Let $B \in \mathcal{B}$. If $\delta \supseteq \{B\}$, then B is g_δ -saturated; so

$$f^{-1}[B] = f^{-1}[g_\delta^{-1}[g_\delta[B]]] = [g_\delta \circ f]^{-1}[g_\delta[B]],$$

a closed subset of X . Thus f is continuous. Suppose f fails to be surjective. Then, because f is continuous, we have disjoint nonempty $B, C \in \mathcal{B}$ with $f[X] \subseteq B$. Pick $\delta \supseteq \{B, C\}$. Then both B and C are g_δ -saturated; hence $g_\delta[B]$, and $g_\delta[C]$ are nonempty and disjoint. But then $g_\delta \circ f$ fails to be surjective. This establishes \mathbf{A}_0 .

In the sequel we fix $\alpha < \omega$, and prove the assertion $\mathbf{A}_{\alpha+1}$.

Let Δ be the set of finite subsets of \mathcal{B} . Using 2.2, there is a single ultrafilter \mathcal{D} on a set I that may be used to witness the hypothesis of $\mathbf{A}_{\alpha+1}$. To be precise, for each $\delta \in \Delta$, the mapping diagram \mathbf{D}_δ consists of maps $p_\delta, h_\delta, k_\delta$, from $Z_\delta I \setminus \mathcal{D}$ to Z_δ, Y , and X respectively, such that p_δ is the codiagonal map (so co-elementary), h_δ and k_δ are each of level $\geq \alpha$, and $p_\delta = g_\delta \circ h_\delta = g_\delta \circ f \circ k_\delta$. To this diagram we adjoin the codiagonal map $q : YI \setminus \mathcal{D} \rightarrow Y$, and define $r := k_\delta \circ (g_\delta I \setminus \mathcal{D}) : YI \setminus \mathcal{D} \rightarrow X$. By 2.4 and 2.5, r is a map of level $\geq \alpha$; we would be done, therefore, if the equality $q = f \circ r$ were true. Not surprisingly, this equality is generally false. What is true are the equalities $g_\delta \circ q = g_\delta \circ f \circ r$. To take advantage of this, we form an ‘‘ultracoproduct’’ of the diagrams \mathbf{D}_δ .

For each $\delta \in \Delta$, let $\hat{\delta} := \{\gamma \in \Delta : \delta \subseteq \gamma\}$. Then the set $\{\hat{\delta} : \delta \in \Delta\}$ clearly satisfies the finite intersection property, and hence extends to an ultrafilter \mathcal{H} on Δ . Form the ‘‘ \mathcal{H} -ultracoproduct’’ diagram \mathbf{D} in the obvious way. Then we have the codiagonal maps $u : X\Delta \setminus \mathcal{H} \rightarrow X$ and $v : Y\Delta \setminus \mathcal{H} \rightarrow Y$. Moreover, again by 2.4 and 2.5, $u \circ (r\Delta \setminus \mathcal{H})$ is of level $\geq \alpha$. We will be done, therefore, once we show that $f \circ u \circ (r\Delta \setminus \mathcal{H}) = v \circ (q\Delta \setminus \mathcal{H})$.

Now the map on the left is just $v \circ ((f \circ r)\Delta \setminus \mathcal{H})$. Suppose $x \in (YI \setminus \mathcal{D})\Delta \setminus \mathcal{H}$ is sent to y_1 under the left map and to y_2 under the right. Let $y'_1 := [(f \circ r)\Delta \setminus \mathcal{H}](x)$ and $y'_2 := [q\Delta \setminus \mathcal{H}](x)$. Then

$$\left[\sum_{\mathcal{H}} g_\delta \right] (y'_1) = \left[\sum_{\mathcal{H}} g_\delta \right] (y'_2).$$

Assume $y_1 \neq y_2$. Then, by the nature of codiagonal maps, there exist disjoint $B_1, B_2 \in \mathcal{B}$, containing y_1 and y_2 in their respective interiors, such that $B_1^\Delta / \mathcal{H} \in y'_1$ and $B_2^\Delta / \mathcal{H} \in y'_2$. If $\delta \supseteq \{B_1, B_2\}$, then both B_1 and B_2 are g_δ -saturated. Thus

$$\{ \delta \in \Delta : g_\delta[B_1] \cap g_\delta[B_2] = \emptyset \} \in \mathcal{H};$$

hence $\prod_{\mathcal{H}} g_\delta[B_1]$ and $\prod_{\mathcal{H}} g_\delta[B_2]$ are disjoint subsets of $\prod_{\mathcal{H}} F(Z_\delta)$. Now

$$\prod_{\mathcal{H}} g_\delta[B_1] \in \left[\sum_{\mathcal{H}} g_\delta \right] (y'_1) \quad \text{and} \quad \prod_{\mathcal{H}} g_\delta[B_2] \in \left[\sum_{\mathcal{H}} g_\delta \right] (y'_2);$$

from which we conclude that

$$\left[\sum_{\mathcal{H}} g_\delta \right] (y'_1) \neq \left[\sum_{\mathcal{H}} g_\delta \right] (y'_2).$$

This contradiction tells us that $y_1 = y_2$ after all, completing the proof. ⊖

We can now give a new proof of the co-elementary chains theorem (Theorem 4.2 in [8]), one where no Banach model theory is used.

THEOREM 3.2. *Let*

$$\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$$

be a co-elementary chain of compacta, with inverse limit X . Then the limit maps $g_n : X \rightarrow X_n, n < \omega$, are all co-elementary.

PROOF. We first prove a weak version of the theorem. This version appears as Proposition 4.1 in [8]. Let

$$\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$$

be a co-elementary chain of compacta. Then there exists a compactum Y and co-elementary maps $h_n : Y \rightarrow X_n, n < \omega$, such that all the equalities $h_n = f_n \circ h_{n+1}$ hold. The proof of this is quite easy, and we repeat it here for the sake of completeness.

By 2.2, there is an ultrafilter \mathcal{D} on a set I and homeomorphisms $k_n : X_{n+1}I \setminus \mathcal{D} \rightarrow X_nI \setminus \mathcal{D}, n < \omega$, such that all the equalities $p_n \circ k_n = f_n \circ p_{n+1}$ hold (where the maps p_n are the obvious codiagonals). Let Y be the inverse limit of this system, with limit maps $j_n : Y \rightarrow X_nI \setminus \mathcal{D}$. Since each k_n is a homeomorphism, so is each j_n , and we set $h_n := p_n \circ j_n$, a co-elementary map. Clearly $f_n \circ h_{n+1} = h_n$ always holds, and there is a map $h : Y \rightarrow X$, uniquely defined by the equalities $g_n \circ h = h_n$.

Now consider the chain of embeddings

$$\langle F(X_n) \xrightarrow{f_n^F} F(X_{n+1}) : n < \omega \rangle,$$

with direct limit \mathcal{A} , and limit embeddings $r_n : F(X_n) \rightarrow \mathcal{A}$. Then (see the argument in 2.9) we may treat X as $S(\mathcal{A})$ and each g_n as r_n^S . (Note: we cannot hope for these

embeddings to be elementary.) For each finite $\delta \subseteq \mathcal{A}$, there is a least $n_\delta < \omega$ such that each member of δ is in the range of r_n for $n \geq n_\delta$. This tells us that X has a lattice base \mathcal{A} such that for each finite $\delta \subseteq \mathcal{A}$ and each $A \in \delta$, A is g_{n_δ} -saturated. This puts us in a position to use 3.1.

We prove that each g_n is of level $\geq \alpha$, for $\alpha < \omega$, by induction on α . Clearly each g_n is of level ≥ 0 ; so assume each g_n to be of fixed level $\geq \alpha$. Then, by 3.1, h is of level $\geq \alpha$ too. Since each h_n is co-elementary, we have now a witness to the fact that each g_n is of level $\geq \alpha + 1$. Thus each g_n is of level $\geq \omega$, and is hence co-elementary by 2.10. \dashv

We had originally thought that 3.1 could be used to prove the α -chains theorem in general, but were unable to get our idea to work. What is missing is a weak version of the assertion, namely the existence of a compactum Y and maps $h_n: Y \rightarrow X_n$ of level $\geq \alpha$ such that all the equalities $h_n = f_n \circ h_{n+1}$ hold. If we could do this, then we could prove the strong version by induction on finite α : The $\alpha = 0$ case is known; assuming the assertion true for fixed α , and that we are given an $(\alpha + 1)$ -chain, we find our compactum Y and maps h_n , all of level $\geq \alpha + 1$. The maps g_n are of level $\geq \alpha$ by the inductive hypothesis, and we conclude that h is of level $\geq \alpha$, by 3.1. Then each g_n is of level $\geq \alpha + 1$, by 2.7.

Rather than pursue the tack just outlined, we abandon 3.1 in favor of a similar-sounding (but somewhat different) lemma.

LEMMA 3.3. *Let $f: X \rightarrow Y$ be a function between compacta, let α be an ordinal, and let \mathcal{A} be a lattice base for X . Suppose that for each finite $\delta \subseteq \mathcal{A}$ there is a map $g_\delta: X \rightarrow Z_\delta$, of level $\geq \alpha$, and a map $h_\delta: Z_\delta \rightarrow Y$, of level $\geq \alpha + 1$, such that $f = h_\delta \circ g_\delta$, and each member of δ is g_δ -saturated. Then f is a map of level $\geq \alpha + 1$.*

PROOF. Assume that $f: X \rightarrow Y$, \mathcal{A} , and α are fixed, with Δ the set of all finite subsets of \mathcal{A} . The ultrafilter \mathcal{H} on Δ is exactly as in 3.1. For each $\delta \in \Delta$, the diagram \mathbf{D}_δ consists of continuous surjections $g_\delta: X \rightarrow Z_\delta$, $h_\delta: Z_\delta \rightarrow Y$, a codiagonal map $p: YI \setminus \mathcal{D} \rightarrow Y$, and a continuous surjection $k_\delta: YI \setminus \mathcal{D} \rightarrow Z_\delta$. (\mathcal{D} need not depend on δ , by 2.2, but that fact is not essential to the argument.) The maps g_δ and k_δ are of level $\geq \alpha$, and the equalities $f = h_\delta \circ g_\delta$ and $p = h_\delta \circ k_\delta$ both hold.

We form the “ultracoproduct” diagram as in 3.1, adding the codiagonal maps $u: X\Delta \setminus \mathcal{H} \rightarrow X$, $v: Y\Delta \setminus \mathcal{H} \rightarrow Y$, along with our original map f . We then define the relation

$$j := u \circ \left(\sum_{\mathcal{H}} g_\delta \right)^{-1} \circ \left(\sum_{\mathcal{H}} k_\delta \right): (YI \setminus \mathcal{D})\Delta \setminus \mathcal{H} \rightarrow X.$$

Once we show j is a map of level $\geq \alpha$, and that $f \circ j = v \circ (p\Delta \setminus \mathcal{H})$, we will have a witness to the fact that f is of level $\geq \alpha + 1$.

To show j is a function, it suffices to show that the kernel of $\sum_{\mathcal{H}} g_\delta$ is contained within the kernel of u . Indeed, suppose $x_1, x_2 \in X\Delta \setminus \mathcal{H}$ are such that $u(x_1) \neq u(x_2)$. Then there are disjoint $A_1, A_2 \in \mathcal{A}$, containing $u(x_1)$ and $u(x_2)$ in their respective interiors, such that $A_1^\Delta / \mathcal{H} \in x_1$ and $A_2^\Delta / \mathcal{H} \in x_2$. If $\delta \supseteq \{A_1, A_2\}$, then both A_1 and A_2 are g_δ -saturated, so

$$\{ \delta \in \Delta : g_\delta[A_1] \cap g_\delta[A_2] = \emptyset \} \in \mathcal{H}.$$

Thus $\prod_{\mathcal{H}} g_\delta[A_1]$ and $\prod_{\mathcal{H}} g_\delta[A_2]$ are disjoint subsets of $\prod_{\mathcal{H}} F(Z_\delta)$, and are elements of $[\sum_{\mathcal{H}} g_\delta](x_1)$ and $[\sum_{\mathcal{H}} g_\delta](x_2)$, respectively. Thus

$$\left[\sum_{\mathcal{H}} g_\delta \right](x_1) \neq \left[\sum_{\mathcal{H}} g_\delta \right](x_2),$$

so j is a function. That j is surjective is clear; that $f \circ j = v \circ (p\Delta) \setminus \mathcal{H}$ is a simple diagram chase. Since

$$j^{-1} = \left(\sum_{\mathcal{H}} k_\delta \right)^{-1} \circ \left(\sum_{\mathcal{H}} g_\delta \right) \circ u^{-1},$$

and $\sum_{\mathcal{H}} g_\delta$ is a closed map, we conclude that j is continuous. Now $\sum_{\mathcal{H}} k_\delta$ and $\sum_{\mathcal{H}} g_\delta$ are maps of level $\geq \alpha$, by 2.4, and $u \circ (\sum_{\mathcal{H}} g_\delta)^{-1}$ is of level $\geq \alpha + 1$, by 2.7. Thus j is of level $\geq \alpha$, by 2.5. -1

We are now ready to establish the α -chains theorem in general.

THEOREM 3.4. *Let α be a fixed ordinal, and let*

$$\langle X_n \xleftarrow{f_n} X_{n+1} : n < \omega \rangle$$

be an α -chain of compacta, with inverse limit X . Then the limit maps $g_n : X \rightarrow X_n$, $n < \omega$, are all of level $\geq \alpha$.

PROOF. Use induction on α . As mentioned above, we need only consider finite α , and the $\alpha = 0$ case is an easy exercise. So assume the α -chains theorem to be true for some fixed α , and let

$$\langle X_l \xleftarrow{f_l} X_{l+1} : l < \omega \rangle$$

be an $(\alpha + 1)$ -chain. Fix $n < \omega$. With the aim of applying 3.3, Y is X_n , and f is g_n . As in the proof of 3.2, \mathcal{A} is the direct limit of the system

$$\langle F(X_l) \xrightarrow{f_l^f} F(X_{l+1}) : l < \omega \rangle$$

of normal disjunctive lattices. Given finite $\delta \subseteq \mathcal{A}$, there is some (least) $m > n$ such that each member of δ is g_m -saturated. Let Z_δ and g_δ be X_m and g_m , respectively, with h_δ the obvious finite composition of the maps f_k , as k runs from n to $m - 1$. g_δ is of level $\geq \alpha$ by our induction hypothesis; h_δ is of level $\geq \alpha + 1$ by 2.5. By 3.3, then, g_n is a map of level $\geq \alpha + 1$. -1

§4. When levels collapse. Here we address the issue of when there is a collapsing of levels of maps between classes of compacta. Let \mathbf{K} and \mathbf{L} be subclasses of \mathbf{CH} , and define $\text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{L})$ to be the class of maps of level $\geq \alpha$, with domains in \mathbf{K} and ranges in \mathbf{L} . (If one of the classes happens to be a single homeomorphism type, say \mathbf{K} is the homeomorphism type of X , then we write $\text{Lev}_{\geq \alpha}(X, \mathbf{L})$ to simplify notation, etc.) Recall that a class \mathbf{K} is a *co-elementary class* if \mathbf{K} is closed under ultracoproducts and co-elementary equivalence. (Of course, being closed under co-elementary equivalence is tantamount to being closed under ultracopowers and co-elementary images; so we could replace the criteria for being a co-elementary class with the conditions of being closed under ultracoproducts and co-elementary images.)

The first result of this section is reminiscent of Robinson’s test from model theory, and its proof is very similar to that of 2.10.

THEOREM 4.1. *Suppose \mathbf{K} and \mathbf{L} are closed under ultracopowers, that $\alpha < \omega$, and that*

$$\text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{L}) = \text{Lev}_{\geq \alpha+1}(\mathbf{K}, \mathbf{L}) \quad \text{and} \quad \text{Lev}_{\geq \alpha}(\mathbf{L}, \mathbf{K}) = \text{Lev}_{\geq \alpha+1}(\mathbf{L}, \mathbf{K}).$$

Then

$$\text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{L}) = \text{Lev}_{\geq \omega}(\mathbf{K}, \mathbf{L}) \quad \text{and} \quad \text{Lev}_{\geq \alpha}(\mathbf{L}, \mathbf{K}) = \text{Lev}_{\geq \omega}(\mathbf{L}, \mathbf{K}).$$

PROOF. Let $f_0: X_0 \rightarrow Y_0$ be a map of level $\geq \alpha$ from a member of \mathbf{K} to a member of \mathbf{L} . Then we build a “co-elementary ladder,” similar to the one in the proof of 2.10, as follows:

Since f_0 is also of level $\geq \alpha + 1$, there are maps $g_0: Y_1 \rightarrow Y_0$ and $h_0: Y_1 \rightarrow X_0$ such that g_0 is co-elementary, h_0 is of level $\geq \alpha$, and $g_0 = f_0 \circ h_0$. Moreover, we may (and do) take g_0 to be an ultracopower codiagonal map; so, in particular, $Y_1 \in \mathbf{L}$, and h_0 is of level $\geq \alpha + 1$. Thus we have maps $j_0: X_1 \rightarrow X_0$ and $f_1: X_1 \rightarrow Y_1$ such that j_0 is co-elementary, f_1 is of level $\geq \alpha$, and $j_0 = h_0 \circ f_1$. As before, we take j_0 to be an ultracopower codiagonal map, so $X_1 \in \mathbf{K}$. This completes the first “rung” of the ladder, and we repeat the process for the map $f_1: X_1 \rightarrow Y_1$, a map of level $\geq \alpha + 1$.

The rest of the proof proceeds exactly like the proof of 2.10, and we conclude that f_0 is co-elementary. –1

With the aid of 3.1, 4.1 has some interesting variations. We first restate what in [8] we call the “sharper” Löwenheim-Skolem theorem. In the sequel, $w(X)$ stands for the weight of a space X .

THEOREM 4.2 (Theorem 3.1 of [8]). *Let $f: X \rightarrow Y$ be a continuous surjection between compacta, with κ an infinite cardinal such that $w(Y) \leq \kappa \leq w(X)$. Then there is a compactum Z and continuous surjections $g: X \rightarrow Z$, $h: Z \rightarrow Y$ such that $w(A) = \kappa$, g is a co-elementary map, and $f = h \circ g$.*

We next bring 3.1 into the picture with the following strengthening of Theorem 3.3 in [8].

THEOREM 4.3. *Let $f: X \rightarrow Y$ be a function between compacta, let α be an ordinal, and let $\kappa \leq w(Y)$ be an infinite cardinal. Suppose that for each compactum Z of weight κ , and each co-elementary map $g: Y \rightarrow Z$, the composition $g \circ f$ is a map of level $\geq \alpha$. Then f is a map of level $\geq \alpha$.*

PROOF. We let Δ be the set of finite subsets of $F(Y)$. For each $\delta \subseteq \Delta$ there is a countable elementary sublattice \mathcal{A}_δ of $F(Y)$, with $\delta \subseteq \mathcal{A}_\delta$. Let $W_\delta := S(\mathcal{A}_\delta)$ (a space of weight \aleph_0), with $r_\delta: Y \rightarrow W_\delta$ denoting the co-elementary map that arises from the inclusion $\mathcal{A}_\delta \subseteq F(Y)$. Then every member of δ is r_δ -saturated. By 4.2, there is a compactum Z_δ of weight κ , and continuous surjections $g_\delta: Y \rightarrow Z_\delta$, $t_\delta: Z_\delta \rightarrow W_\delta$, such that g_δ is co-elementary and $r_\delta = t_\delta \circ g_\delta$. So each g_δ is a co-elementary map onto a compactum of weight κ ; by hypothesis, then, $g_\delta \circ f$ must be a map of level $\geq \alpha$. By 3.1, f must be a map of level $\geq \alpha$. –1

For any class \mathbf{K} and cardinal κ , let

$$\mathbf{K}_\kappa := \{ X \in \mathbf{K} : w(X) = \kappa \}.$$

The following is a variation (though not, strictly speaking, an improvement) on 4.1.

THEOREM 4.4. *Suppose \mathbf{K} and \mathbf{L} are closed under ultracopowers, as well as co-elementary images, that $0 < \alpha < \omega$, and, for some infinite cardinal κ , that*

$$\text{Lev}_{\geq\alpha}(\mathbf{K}_\kappa, \mathbf{L}_\kappa) = \text{Lev}_{\geq\alpha+1}(\mathbf{K}_\kappa, \mathbf{L}_\kappa) \quad \text{and} \quad \text{Lev}_{\geq\alpha}(\mathbf{L}_\kappa, \mathbf{K}_\kappa) = \text{Lev}_{\geq\alpha+1}(\mathbf{L}_\kappa, \mathbf{K}_\kappa).$$

Then

$$\text{Lev}_{\geq\alpha}(\mathbf{K}, \mathbf{L}) = \text{Lev}_{\geq\omega}(\mathbf{K}, \mathbf{L}) \quad \text{and} \quad \text{Lev}_{\geq\alpha}(\mathbf{L}, \mathbf{K}) = \text{Lev}_{\geq\omega}(\mathbf{L}, \mathbf{K}).$$

(The assertion also holds in the case $\alpha = 0$, if we assume that neither \mathbf{K} nor \mathbf{L} contains any finite spaces.)

PROOF. Let $f : X \rightarrow Y$ be a map of level $\geq \alpha$, between members of \mathbf{K} and \mathbf{L} respectively. By 4.1, it suffices to show that f is of level $\geq \alpha + 1$. Assume first that $\kappa \leq w(Y)$. By 4.3, it suffices to show that for each compactum Z of weight κ and each co-elementary map $g : Y \rightarrow Z$, we have that $g \circ f$ is of level $\geq \alpha + 1$. So let $g : Y \rightarrow Z$ be given. By 4.2, there is a factorization $u : X \rightarrow W$, $v : W \rightarrow Z$ such that u is co-elementary, $w(W) = w(Z) = \kappa$, and $g \circ f = v \circ u$. Now $W \in \mathbf{K}_\kappa$ and $Z \in \mathbf{L}_\kappa$, and $g \circ f$ is of level $\geq \alpha$. Thus, by 2.7, v is also of level $\geq \alpha$. By hypothesis, v is of level $\geq \alpha + 1$; consequently, so is $g \circ f$.

If $\kappa > w(Y)$, and we are dealing with the case $\alpha > 0$, then we must consider the possibility that Y is finite. But f is a co-existential map, and hence clearly a bijection (i.e., a homeomorphism) in that situation. So we may as well assume that Y is infinite. If we are dealing with the case $\alpha = 0$, then we take Y to be infinite by fiat.

That said, we find an ultrafilter \mathcal{D} on a set I such that $w(YI \setminus \mathcal{D}) \geq \kappa$ (see [2]). By the argument in the first paragraph, since both \mathbf{K} and \mathbf{L} are closed under ultracopowers, we conclude that $fI \setminus \mathcal{D}$ is of level $\geq \alpha + 1$. From our work in §2, we infer that f is of level $\geq \alpha + 1$ too. ⊔

Given an ordinal α , we say $X \in \mathbf{K}$ is α -closed in \mathbf{K} if $\text{Lev}_{\geq 0}(\mathbf{K}, X) = \text{Lev}_{\geq\alpha}(\mathbf{K}, X)$. (1-closed = co-existentially closed [8].) Define

$$\mathbf{K}^\alpha := \{ X \in \mathbf{K} : X \text{ is } \alpha\text{-closed in } \mathbf{K} \}.$$

Using the technique of “adding roots,” by which one produces algebraically closed extensions of fields (and, more generally, existentially closed extensions of models of a universal-existential theory), we showed (Theorem 6.1 in [8]) that if \mathbf{K} is a co-elementary class that is *co-inductive*, i.e., closed under limits of 0-chains, and if $X \in \mathbf{K}$ is infinite, then there is a compactum $Y \in \mathbf{K}^1$, of the same weight as X , such that X is a continuous image of Y . (So \mathbf{K}^1 is quite substantial under these circumstances.) **CH**, **BS**, and **CON** (the class of *continua*, i.e., connected compacta) are easily seen to be examples of co-inductive co-elementary classes. In [8] we showed $\mathbf{CH}^1 = \mathbf{BS}^1 = \{\text{Boolean spaces without isolated points}\}$ (Proposition 6.2), and that every member of \mathbf{CON}^1 (i.e., every *co-existentially closed continuum*) is *indecomposable*, i.e., incapable of being written as the union of two proper subcontinua (Proposition 6.3). We posed the question of whether \mathbf{CON}^1 is a co-elementary class, and conjectured that every co-existentially closed continuum is of (Lebesgue covering) dimension one. While the question of co-elementarity is still open, we

have been able to settle the conjecture in the affirmative. We are grateful to Wayne Lewis [17], who suggested the use of a theorem of D. C. Wilson [25].

THEOREM 4.5. *Every co-existentially closed continuum is an indecomposable continuum of dimension one.*

PROOF. Because of Proposition 6.3 of [8], we need only concentrate on the issue of dimension.

Let Q denote the Hilbert cube, the usual topological product of countably many copies of the closed unit interval. It is well known [24] that every metrizable compactum can be replicated as a (closed) subspace of Q . Next, let M denote the Menger universal curve, a one-dimensional Peano (i.e., locally connected metrizable) continuum. Perhaps less well known is the fact [18] that every one-dimensional metrizable compactum can be replicated as a (closed) subspace of M . Wilson’s theorem [25] says that there is a continuous surjection $f : M \rightarrow Q$ whose point-inverses are all homeomorphic to M . So f is, in particular, monotone; hence inverse images of subcontinua of Q are subcontinua of M . Now let X be any metrizable continuum, viewed as a subspace of Q . Then $f^{-1}[X]$ is a subcontinuum of M that maps via f onto X . Since M is one-dimensional, so is $f^{-1}[X]$.

So we know that every metrizable continuum is a continuous image of a metrizable continuum that is one-dimensional. Let X now be an arbitrary continuum. Then, by Löwenheim-Skolem, there is a co-elementary map $f : X \rightarrow Y$, where Y is a metrizable continuum. Using the result in the preceding paragraph, let $g : Z \rightarrow Y$ be a continuous surjection, where Z is a metrizable continuum of dimension one. Because of the co-elementarity of f , there is a homeomorphism $h : XI \setminus \mathcal{D} \rightarrow YI \setminus \mathcal{D}$ of ultracopowers such that $f \circ p = q \circ h$, where p and q are the obvious codiagonal maps. Since covering dimension is an invariant of co-elementary equivalence [2], we know that $ZI \setminus \mathcal{D}$ is a continuum of dimension one. Thus $p \circ h^{-1} \circ (gI \setminus \mathcal{D})$ is a continuous surjection from a continuum of dimension one onto X .

Now suppose X is 1-closed in **CON**. Then, by the paragraph above, there is a continuous surjection $f : Y \rightarrow X$, where Y is a continuum of dimension one. But f is a co-existential map, and co-existential maps preserve being infinite, and cannot raise dimension. The dimension of X cannot be zero; hence it must be one. \dashv

The following result records some general information concerning levels of maps between classes, and is an easy corollary of the general results above.

COROLLARY 4.6. *Let \mathbf{K} be a class of compacta, α an ordinal.*

(i) *Suppose $\alpha > 0$, and \mathbf{K}^α is closed under ultracopowers. Then*

$$\text{Lev}_{\geq 0}(\mathbf{K}^\alpha, \mathbf{K}^\alpha) = \text{Lev}_{\geq \omega}(\mathbf{K}^\alpha, \mathbf{K}^\alpha).$$

(ii) *Suppose \mathbf{K} is closed under ultracopowers. Then*

$$\text{Lev}_{\geq \alpha}(\mathbf{K}^\alpha, \mathbf{K}) = \text{Lev}_{\geq \alpha}(\mathbf{K}^\alpha, \mathbf{K}^\alpha).$$

(iii) *Suppose \mathbf{K} is closed under ultracopowers, and*

$$\text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{CH}) = \text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{K}).$$

Then

$$\text{Lev}_{\geq \alpha+1}(\mathbf{K}', \mathbf{CH}) = \text{Lev}_{\geq \alpha+1}(\mathbf{K}', \mathbf{K}')$$

(where $\mathbf{K}' := \mathbf{CH} \setminus \mathbf{K}$).

PROOF.

Ad (i): By definition of \mathbf{K}^α , $\text{Lev}_{\geq 0}(\mathbf{K}, \mathbf{K}^\alpha) = \text{Lev}_{\geq \alpha}(\mathbf{K}, \mathbf{K}^\alpha)$. The conclusion is immediate, by 4.1.

Ad (ii): There is nothing to prove if $\alpha = 0$. So assume $\alpha > 0$, and suppose X is α -closed in \mathbf{K} , $Y \in \mathbf{K}$, and $f: X \rightarrow Y$ is of level $\geq \alpha$. Let $p: YI \setminus \mathcal{D} \rightarrow Y$ and $g: YI \setminus \mathcal{D} \rightarrow X$ witness the fact; i.e., p is a codiagonal map, g is of level $\geq \alpha - 1$, and $f \circ g = p$. Let $Z \in \mathbf{K}$, with $h: Z \rightarrow Y$ a continuous surjection. Let $q: ZI \setminus \mathcal{D} \rightarrow Z$ be the appropriate codiagonal map. Since \mathbf{K} is closed under ultracopowers, and $X \in \mathbf{K}^\alpha$, we know that both g and $g \circ (hI \setminus \mathcal{D})$ are of level $\geq \alpha$. Then $f \circ g \circ (hI \setminus \mathcal{D}) = p \circ (hI \setminus \mathcal{D}) = h \circ q$ is of level $\geq \alpha$, by 2.5. By 2.7, h is also of level $\geq \alpha$; hence $Y \in \mathbf{K}^\alpha$.

Ad (iii): Suppose $f: X \rightarrow Y$ is a map of level $\geq \alpha + 1$, and $Y \in \mathbf{K}$. We need to show $X \in \mathbf{K}$. But this is immediate from the definition of level, plus our hypotheses. -1

REMARK 4.7. We have very few results concerning the nature of \mathbf{K}^α , given information about \mathbf{K} . We can prove quite easily, though, that \mathbf{CH}^2 , \mathbf{BS}^2 , and \mathbf{CON}^2 are all empty. Indeed, let X be any compactum, with Y the disjoint union of X with a singleton, and Z the product of X with a Cantor discontinuum. Then there exist continuous surjections $f: Y \rightarrow X$ and $g: Z \rightarrow X$. Assume X is now 2-closed in \mathbf{CH} . Maps of level ≥ 1 preserve the property of having no isolated points (Proposition 2.8 in [8]); so we conclude that X has no isolated points because Z has none. On the other hand, since the class of compacta without isolated points is co-elementary, and f is of level ≥ 2 , we conclude, by 4.6 (iii), that X has an isolated point because Y does. Thus \mathbf{CH}^2 is empty. If X above happens to be Boolean, so are Y and Z ; hence the same argument shows that \mathbf{BS}^2 is empty. Now assume $X \in \mathbf{CON}^2$. Then X has dimension one, by 4.5. Let Y be the product of X with the Hilbert cube. Then there is a continuous surjection $f: Y \rightarrow X$, and Y is an infinite-dimensional continuum. Since the class of finite-dimensional continua is co-elementary, as well as closed under images of maps of level ≥ 1 (Proposition 2.6 in [8]), and f is of level ≥ 2 , we conclude, again by 4.6 (iii), that X is infinite-dimensional because Y is. Thus \mathbf{CON}^2 is empty.

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